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Rotation of the deformable Earth with the viscous fluid core

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Г. А. Красинский. Вращение деформируемой Земли с вязким жидким ядром.

Ключевые слова: Вращение Земли, геодинамика, приливы.

Дается вывод дифференциальных уравнений вращения неупругой Земли с вязким жидким ядром, которые затем используются для интерпретации ряда наблюдаемых эффектов во вращении Земли, в приливных вариациях положения наблюдателя и в приливных вариациях геопотенциала. Уравнения выведены с использованием метода Пуанкаре для моделирования динамических эффектов жидкого ядра и метода Сасао для моделирования взаимодействия жидкого ядра и мантии в терминах динамического числа Лава k_2^d. Уравнения учитывают ряд эффектов, игнорируемых в принятой модели вращения Земли, а именно, возмущающих моментов, обусловленных взаимодействием деформируемой приливами Земли с приливообразующими небесными телами (включая диссипативный кросс-эффект взаимодействия с Солнцем лунных приливов и солнечных — с Луной). Возмущения этого типа не могут быть учтены в рамках принятой теории вращения Земли, поскольку они не описываются методом передаточной функции, использованным при построении этой теории. Выведенные уравнения явным образом зависят от двух параметров, описывающих диссипативные эффекты во вращении Земли. Этими параметрами являются эффективный фазовый лаг δ приливов, моделирующий диссипацию Земли в целом, и фазовый лаг δ_c, характиризующий диссипацию в жидком ядре. В настоящее время численное значение параметра δ может считаться известным с достаточной точностью из анализа лунных лазерных измерений и поэтому δ_c является единственным параметром, который можно варьировать для теоретической интерпретации всего многообразия наблюдаемых эффектов диссипации во вращении Земли. Проведенный предварительный анализ показал, что наиболее заметные наблюдаемые диссипативные эффекты — эксцесс наблюденного векового изменения наклона экватора к эклиптике по сравнению с теоретическим значением для твердотельной модели и значительные квадратурные составляющие 18.6 летней и полугодичной нутации — вызваны совместным влиянием обоих упомянутых диссипативных параметров. При учете влияния океанических приливов на квадратурные амплитуды оценки величины δ_c, выведенные из анализа этих амплитуд и векового изменения наклона экватора к эклиптике, становятся согласующимися, давая значения из интервала $\delta_c \approx 0.1 - 0.2$. Действие параметра δ_c на квадратурные составляющие нутаций осуществляется через коэффициент затухания свободной нутации жидкого ядра, что подтверждает предположение работы (Dehant & Defraign, 1997, [2]) о важной роли данного коэффициента для объяснения наблюдаемых больших значений этих амплитуд. Значительная величина приливного лага для жидкого ядра δ_c свидетельствует о высокой вязкости жидкого ядра Земли, и, как следствие, о том, что энергия вращения Земли диссипируется, в основном, жидким ядром.

G. A. Krasinsky. Rotation of the deformable Earth with the viscous fluid core.

Keywords: Rotation of the Earth, Geodynamics, Tides.

Differential equations of rotation of the Earth with the dissipative fluid core are developed and applied to explanation of a number of observed effects in the Earth's rotation, as well as in the tidal site displacements and in the tidal variations of the geopotential. The equations are derived by Poincare's method to model the dynamical effects of the fluid core, and by Sasao's formulation to model interaction of the fluid core with the non-rigid mantle in terms of the dynamic Love number k_2^d. The equations take into account some effects ignored in the conventional theory of the Earth's rotation, namely caused by the perturbing torques due to interaction of the potentials, induced by the tidal deformations of the Earth and its fluid core, with the tide arousing bodies (including the dissipative cross interaction of the lunar tides with the Sun and the solar tides with the Moon). Perturbations of this kind could not be accounted in the conventional theory in which only those obtainable by the method of the transfer function are considered. The derived equations explicitly depend on two parameters characterizing the dissipation in the Earth's rotation. These parameters are the effective tidal phase lag δ due to the dissipation of energy in the Earth as a whole, and the tidal phase lag δ_c due to the dissipation by the differential rotation of the fluid core. At present the numerical value of δ is well known from analysis of LLR observations and thus there is the single parameter δ_c at disposal for explanation of the all variety of the dissipative effects in the Earth rotation. The preliminary analysis has shown that the most noticeable of such effects — the excess of the observed secular obliquity rate compared with predictions of the rigid body model, and the large out-phase amplitudes of the 18.6-year and semi-annual nutations indeed may be explained as the result of the combined action of these two types of dissipation. Accounting for the impact of the ocean tides on the out-phase amplitudes the estimations of δ_c derived from the obliquity rate and from the out-phase amplitudes become consistent giving $\delta_c \approx 0.1 - 0.2$. The impact of δ_c on the out-phase amplitudes is mainly due to dumping of Free Core Nutation and thus the conjecture made in (Dehant & Defraign, 1997, [2]) on importance of this dumping for explanation of the large observed out-phase amplitudes is justified. The rather large value δ_c evidences the high viscosity of the fluid core. Thus it seems that the main part of the rotational energy of the Earth dissipates in the fluid core.

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1. Introduction

1.1. Motivations

Progress of the VLBI technic has lead to accumulation of quite accurate observed positions of the Celestial Pole providing valuable experimental information for study of the Earth's rotation. To interpret these data a precise dynamical model of the Earth's rotation based on a sound mathematical ground is still in a strong need. Though Nutation IAU2000 recently adopted as an international standard does improve fitting of the observed Earth's orientation parameters, the physics behind this theory is rather uncertain as the theory till now keeps a semi-empirical nature. The prehistory of the present models of rotation of the non-rigid Earth is well known and in brief may be described in the following way (not pretending to an exhausting review of a large number of dedicated works, and referring to the paper (Dehant & Defraign, 1997, [2])). After elucidating that the difference of the observed Chandler's period of the free pole motion with the Euler's period for the rigid body model is caused by elasticity of the Earth, it became clear that a relevant theory of the Earth's rotation must account for the non-rigidity of the Earth. At first it seemed necessary to consider the Earth's rotation in the frame of general equations of the theory of elasticity which are quite complicated being the system with infinite number of degrees of freedom, unlike the simple equations of the classic dynamics. In this way the mathematical models by Molodensky, Jefferys and Wahr had been developed and formed the mathematical base of Nutation 1980. These models account for effects of the Earth's fluid core after the pioneering work by Poincare [13]. Drawback of these theories is not only their complication but also the impossibility to parameterize them in terms of some integral characteristics of the Earth which might be estimated from analysis of observations.

The further significant progress is obliged to the work by Sasao, Okubo, and Saito (Sasao et al, 1980, [15]) where an alternative but equivalent approach has been developed (so called SOS model) in which the Earth's rotation is described by a system of ordinary differential equations generalizing those given by Poincare for the rigid Earth with the fluid core. In this generalization the tidal variations of the moments of inertia of the elastic mantle and of the fluid core are accounted being parameterized with the help of so called compliances κ, γ, β. The compliances κ, γ may be expressed in terms of the static k_2 and dynamic k_2^d Love numbers, relatively. Loosely speaking, the dynamic Love number k_2^d scales perturbations in the moments of inertia of the fluid core caused by tides in the mantle, and vice versa. The compliance β can be expressed through a parameter k_2^c which plays part of the Love number of the fluid core. Its action diminishes the Free Core Nutation frequency in the same way as the Love number k_2 reduces the Euler's frequency of the free pole motion to the Chandler's frequency.

The compliances (or the corresponding Love numbers) may be obtained as theoretically making use of the constants of the up-to-date models of the Earth's interior, as from analysis of VLBI data by fitting the rotation theory to these data. With the help of the SOS model it appeared possible to explain the most noticeable signature in the residuals of the observed Earth's orientation parameters compared with Nutation 1980. This signature is of the annual period resulting from near resonant interaction of the annual harmonics with the 431-day Free Core Nutation caused by the fluid core. In fact the SOS model is the theoretical base of Nutation IERS96 that had ensured the satisfactory fit to VLBI data, and of Nutation 2000 adopted as a standard for astronomical applications. However the SOS model has deficiencies on which the attention seems to be not yet paid:

1. In the rigorous formulation the SOS model describes rotation of the Earth with the elastic mantle and fluid core. In efforts to apply this model for processing the observed parameters of the Earth's rotation, the dissipative effects were treated in a formal way assuming that the compliances have imaginary parts to be estimated from VLBI observations (see Shirai & Fukushima, 2001, [16]). In reality such an approach is equivalent to incorporation of some empirical terms into the differential equations of the SOS model. In fact, in order to describe all the dissipative effects, five imaginary constants are the solve-for parameters of the fitting; and thus the five empirical terms have to be added into the differential equations of the Earth's rotation. Physical meaning of these terms is uncertain and the derived from them value of the tidal phase lag δ (see Shirai & Fukushima, 2001, [16]) appeared to be roughly inconsistent with the reliable LLR estimate of this parameter. The attempt to avoid the empirical approach described in (Dehant & Defraign, 1997, [2]) has shown that the observed out-phase amplitudes of nutation cannot be satisfactory explained on the base of the conventional model if no empirical terms are introduced to it.

2. In the differential equations of the Earth's rotation of the SOS model there are accounted only those perturbing terms from the non-rigidity of the Earth which are generated by the tidal variations of the matrix of inertia, the rigid body approximation still being used to model the perturbing torques. The resulting perturbations can be expressed in terms of the rigid body nutations and thus it appears possible to present them as a linear differential operator applied to the rigid body nutations. In practice this transformation is carried out by transforming each of the nutational harmonics with the help of so called transfer function. However some important perturbations cannot be obtained in this way and indeed they are ignored in the SOS model. In brief their origin may be explained in the following way. The tidally deformed mantle and fluid core, while interacting with the perturbing celestial bodies give rise to additional torques which are proportional to the Love numbers k_2 and k_2^d, relatively. It is well known that for the elastic Earth the torques of the first type vanish; but that is not the case if the effective phase lag δ of the tides is accounted to describe the impact of dissipation of energy by the body tides. The ignored dissipative torques rule evolution of the Earth-Moon system and so are very important. In particular they are responsible for a small (but informative for geophysics) part of the observed obliquity rate unexplainable in the rigid body model. It is easy to see that no secular rate in obliquity in principle can be modeled in the frame of the mentioned above formal considerations of the imaginary parts of parameters of

the SOS model. Other observable effects are caused by the torques (ignored in the conventional model) which are proportional to the dynamical Love number k_2^d and so depend on the differential rotation of the fluid core. These torques generate not only detectable obliquity rate but also energy conserving in-phase perturbations which cannot be neglected, though being small (the additional precession rate is about 10 mas/cy and the in-phase amplitude of 18.6 year nutations is about 20 μas). The ignored torques of this type depend on the Euler angles of nutation and precession in such a way that the perturbations from them are not obtainable by any transfer function.

3. Presented in this paper a preliminary analysis of the observed out-phase amplitudes (given by the contemporary semi-empirical theories fitted to VLBI data), as well as of the observed value of the obliquity rate shows that the fluid core is strongly dissipative. That necessitates to develop the model of the Earth's rotation for the case of the viscous fluid core.

These deficiencies of the adopted theory of the Earth's rotation have motivated this study. The aim of the work is not to present a new completed theory of nutation but revise the SOS equations deriving more rigorous and full differential equations of rotation of the anelastic Earth with the dissipative fluid core. It seems that integration of such equations is reasonable to carry out not by the classic analytical methods of Celestial Mechanics (proved to give too much room for unsound empirical assumptions) but by rigorous numerical integration which prevents such assumptions and could be easily reproduced by any interested researcher. Then the fitting to the accumulating dataset of the observed Earth's orientation parameters could be produced in a regular way (for instance annually) generating new theories of the Earth rotation with improvable values of the geophysical parameters as one of the IERS products.

The rigorous differential equations of rotation of the Earth with the anelastic mantle and viscous fluid core being rather simple demand tedious analytical manipulations to be derived. For a control and in order to make a verification be possible we start from the basic principles of the body tide modeling and present all the analytical transformations in details from the very beginning, at first for the elastic Earth without the fluid core (Section 2) and then in the general case of the anelastic mantle and the dissipative fluid core (Section 3). Hopefully the more easy understanding justifies the repeatings which could not be avoided. To facilitate the usage of the derived differential equations of the Earth's rotation, they are given already in this Introduction (without proof) and compared there with the conventional SOS equations presented in the same notations. In Section 4 the following applications of these equations are studied:

1. The discrepancy between the observed secular rate of the obliquity and its rigid body prediction is explained as caused by the combined action of the tidal phase lags δ and δ_c.

2. The observed out-phase amplitudes of the 18.6-year and semi-annual nutations appear to be possible to interpret in the same way; as the result four independent estimates of δ_c have been obtained.

3. Analytical expression for the variations with the sidereal period of the harmonics c_2^1, s_2^1 of the geopotential (so called K_1 tide) is derived in terms of the dynamical Love number k_2^d. This expression shows that to be consistent with Nutation IAU2000, the amplitude of the K_1 tide has to be set equal to 646×10^{-12} which value significantly differs from the value 471.8×10^{-12} recommended by the IERS standards (McCarthy, 2000, [11]). Our value of the K_1 amplitude has been recently confirmed experimentally by analysis of SLR observations of the satellites Etalon-1,2 (Ivanova & Shuygina, 2003, [4]).

The mathematical content of Sections 2-3 is not obligatory for reading the applications.

Appendices contain some auxiliary considerations of a technical character.

1.2. Notations, constants and auxiliary relations

The variables in use are defined when they arrive at the first time in the text. For convenience in this section all the notations are gathered and the adopted numerical values of a number of the parameters are also given. So the following notations will be used:

$\overline{r} = (x_1, x_2, x_3)$: coordinates of a perturbing body in the Earth fixed equatorial system,
$\overline{\rho} = (\rho_1, \rho_2, \rho_3)$: the unit vector $\overline{\rho} = \overline{r}/r$,
$\overline{r}^e = (x_1^e, x_2^e, x_3^e)$: ecliptic coordinates of the perturbing body in the inertial system,
$\overline{\omega} = (\omega_1, \omega_2, \omega_3)$: angular velocity of the Earth in the coordinate frame fixed to the mantle (Tisserand's axes),
$\overline{v} = (v_1, v_2, v_3)$: angular velocity of the Earth's fluid core; $v_3 = 0$.
$I_0 = (A, A, C),\ A < C$: unperturbed diagonal matrix of inertia,
c^{ik} : tidally induced elements of the matrix I of the moments of inertia,
$I_0^c = (A^c, A^c, C^c),\ A^c < C^d$: unperturbed matrix of inertia of the fluid core,
c_c^{ik} : tidally induced elements of the matrix I^c of inertia of the fluid core,
m_E, R : mass and radius of the Earth,
m : mass of the perturbing body (Moon or Sun),
$g = A/(m_E R^2)$: normalized moment of inertia,
$e = (C - A)/C$: dynamical flattening of the Earth,
$e_c = (C_c - A_c)/C$: dynamical flattening of the fluid core,
$J_2 = eg$: coefficient of the zonal harmonics of geopotential,
$\omega = |\overline{\omega}|$: angular velocity of the Earth,
G : gravitational constant,
θ : angle of nutation,
ϕ : angle of precession,
ψ : rotational angle.

Table 1. Numerical values of some constants.

Symbol	Value	Comment
p_1	3783.88 "/cy	Lunar precession constant
p_2	1737.71 "/cy	Solar precession constant
$\epsilon_1 = p_1/e\omega$	2.46×10^{-5}	Constant in dissipative terms
$\epsilon_2 = p_2/e\omega$	1.13×10^{-5}	Constant in dissipative terms
$\epsilon = (p_1\epsilon_1 + p_2\epsilon_2)/p$	2.04×10^{-5}	Constant in dissipative terms
α	0.11380	A_c/A
e	3.246×10^{-3}	Dynamical flattening
e_c	2.547×10^{-3}	Dynamical flattening of the core
k_s	0.93831	Secular Love number
k_2	0.3003	Static Love number
k_2^d	0.0642	Dynamic Love number
k_2^c	0.0201	Love number of the core
$\sigma = k_2/k_s$	0.3201	Normalized Love number k_2
$\nu = k_2^d/k_s$	0.0684	Normalized Love number k_2^d
$\mu = k_2^c/k_s$	0.0214	Normalized Love number k_2^c
$\kappa = e\sigma$	1.039×10^{-3}	Compliance, the Earth as a whole
$\gamma = e\nu/\alpha$	1.965×10^{-3}	Compliance, mantle-core
$\beta = e\mu/\alpha$	6.94×10^{-4}	Compliance, core
f^{ch}	0.01571 rad/day	Chandler's frequency
f_c	0.01458 rad/day	Free Core Nutation frequency
δ	0.03767	Phase delay of the body tides

Notations κ, γ, β for compliances coincide with those in the most part of other works. We have preferred however to present the derived analytical expressions in terms not of the compliances but of the parameters σ, ν, μ (which are close to the Love numbers k_2, k_2^d, k_2^c).

In the theory of nutation the following complex variables will be used instead of the angular velocities $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\overline{v} = (v_1, v_2, v_3)$:

$$u = \omega_1 + i\omega_2, \; v = v_1 + iv_2,$$

and

$$\xi = \rho_1 + i\rho_2, \; \zeta = \rho_3,$$

for coordinates of the perturbing body (here i is $\sqrt{-1}$).

Let $\overline{R}$ be a vector with coordinates given in the Earth's fixed rotating frame. The following identity holds true

$$\dot{\overline{R}} = \frac{\partial \overline{R}}{\partial t} - \overline{\omega} \times \overline{R}, \tag{1}$$

where the symbol ∂ stands for the time derivative calculated ignoring time-dependence on the rotational angle ψ in the matrix of transformation to the inertial coordinate frame. In particular

$$\dot{\overline{\omega}} = \frac{\partial \overline{\omega}}{\partial t}.$$

1.3. Conventional SOS model of the Earth's rotation

The SOS equations of the Earth's rotation may be written in the following form (see Moritz & Mueller, 1987, [12]):

$$\dot{u} - ie\omega u + \alpha(\dot{v} + i\omega v) + \frac{\omega}{A}(\dot{c} + i\omega c) = L, \tag{2}$$

$$\dot{v} + \dot{u} + i\omega v \frac{C_c}{A_c} + \omega \frac{\dot{c}_c}{A_c} = 0, \tag{3}$$

where L is the rigid body torque normalized dividing by the moment of inertia A,

$$\begin{aligned} u &= \omega_1 + i\omega_2, \\ v &= v_1 + iv_2, \end{aligned}$$

and $c = c^{13} + ic^{23}$, $c_c = c_c^{13} + ic_c^{23}$ are complex combinations of the tidally induced non-diagonal components of the matrices of inertia I, I^c of the Earth and its fluid core, connected with u, v, L by the relations:

$$c = D_{11}\left(u - i\frac{L}{e\omega}\right) + D_{12}v, \tag{4}$$

$$c_c = D_{21}\left(u - i\frac{L}{e\omega}\right) + D_{22}v. \tag{5}$$

In equations (3)–(5) the constant $\alpha = \frac{A_c}{A}$ is the ratio of the main moments of inertia of the fluid core and the Earth as a whole, the coefficients D_{mn} define response of the matrices of inertia of the mantle and fluid core on perturbations from the tides of the two types: aroused by outer bodies or by rotation of the Earth as a whole, and by the differential rotation of its fluid core. Numerical values of the integral geophysical constants D_{mn} are provided by models of the Earth's interior. At present the PREM model (Dziewonsky & Andersen, 1987 [3]) is considered to be the most accurate. The important theoretical identity holds true:

$$D_{21} = D_{12}. \tag{6}$$

The normalized rigid body torque L in equation (2) is given by the expression

$$L = -ip\omega\xi\zeta,$$

where p is the parameter of the lunar or solar precession

$$p = \frac{3}{2}\frac{mG}{r^3}e, \tag{7}$$

$\xi = \rho_1 + i\rho_2,\ \zeta = \rho_3$ are coordinates of the tide arousing body, and e is the dynamical flattening.

In fact the rigid body torque is the sum of the lunar and solar components $L = L^1 + L^2$ where $L^k = -ip_k\xi^k\zeta^k,\ p = p_1 + p_2,\ p_1, p_2$ being parameters of the lunar and solar precession. The explicit form of dependence of the torque L on the coordinates is of no importance in the conventional theory of rotation of the non-rigid Earth where only the perturbations obtainable by the method of transfer function are taken into consideration. However that is not the case for the more refined dynamical model of the Earth's rotation.

If numerical values of the geophysical constants D_{mn} are known, the equations of motion of the deformable Earth with the fluid core may be obtained by the method of Poincare applying the basic dynamical principles (see Moritz & Mueller, 1987, [12]) without any further geophysical considerations. Equations (2)–(5) have been deduced in this way but with some simplifications that bring to minor errors which are noticeable on the contemporary level of the accuracy. In the following sections more rigorous equations of the Earth's rotation are developed in detail and compared with conventional equations (2)–(5).

The coefficients $D_{11},\ D_{12}$ may be expressed in terms of the static and dynamic Love numbers k_2, k_2^d:

$$\frac{D_{11}}{A} = \frac{1}{3G}\frac{R^5}{A}\omega k_2, \tag{8}$$

$$\frac{D_{12}}{A} = \frac{1}{3G}\frac{R^5}{A}\omega k_2^d. \tag{9}$$

The coefficients D_{22} may be formally presented in analogous way

$$\frac{D_{22}}{A} = \frac{1}{3G}\frac{R^5}{A}\omega k_2^c, \tag{10}$$

where the undimensional constant k_2^c is responsible for perturbations of the matrix of inertia of the fluid core caused by its differential rotation, and will be referred as the Love number of the fluid core.

Note that relation (9) differs from that in (Moritz & Mueller, 1987, [12]) by the sign. Our definition of the dynamic Love numbers k_2^d corresponds to positive value of this constant that seems more natural and convenient. Defining undimensional parameters σ, ν, μ by the relation:

$$\sigma = \frac{R^3\omega^2 k_2}{3Gm_E J_2} \approx \frac{1}{3}, \nu = \sigma\frac{k_2^d}{k_2} \approx \frac{1}{15}, \;\; \mu = \sigma\frac{k_2^c}{k_2} \approx \frac{1}{45}, \tag{11}$$

the expressions for the coefficients D_{11}, D_{12} may be rewritten in the following simple form:

$$\frac{D_{11}}{A} = \frac{\sigma}{\omega}e, \tag{12}$$

$$\frac{D_{12}}{A} = \frac{\nu}{\omega}e, \tag{13}$$

$$\frac{D_{22}}{A} = \frac{\mu}{\omega}e. \tag{14}$$

Instead of the coefficients σ, ν, μ so called compliances τ, γ, β are commonly used. They may be introduced by the relations:

$$\kappa = e\sigma, \;\; \gamma = e\frac{\nu}{\alpha}, \;\; \beta = e\frac{\mu}{\alpha}.$$

In terms of the 'secular' Love number k_s defined by the expression

$$k_s = \frac{3Gm_E J_2}{R^3\omega^2} \approx 0.93831 \tag{15}$$

the parameters σ, ν, μ may be presented by the relations

$$\sigma = \frac{k_2}{k_s}, \nu = \frac{k_2^d}{k_s}, \;\; \mu = \frac{k_2^c}{k_s}. \tag{16}$$

In these notations SOS equations have the following form:

$$\dot{u}(1+e\sigma) - ie\omega(1-\sigma)u + (\alpha + e\nu)(\dot{v} + i\omega v) = L + i\frac{\sigma}{\omega}\frac{\partial L}{\partial t}, \tag{17}$$

$$\dot{u} + \dot{v} + iv\omega\left(1 + e_c - \mu\frac{e}{\alpha}\right) = \frac{\nu}{\alpha}\left[L\,(1-e) - \frac{i}{\omega}\frac{\partial L}{\partial t}\right]. \tag{18}$$

The normalized perturbing torque L implicitly depends on the three Euler's angles: the nutation angle θ, the angle of precession ϕ, and the rotational angle ψ. More exactly, L depends on the coordinates $\overline{r} = (r_1, r_2, r_3)$ of the perturbing body in the rotating frame which have to be expressed through coordinates of this body $\overline{r}^e = (r_1^e, r_2^e, r_3^e)$ in the inertial ecliptical frame:

$$\overline{r} = P_3(\psi)P_1(\theta)P_3(\phi)\overline{r}^e,$$

where P_1, P_3 are rotational matrices relatively to the first and third coordinate axes. In this way the explicit dependence of the torque L on the Euler's angles may be obtained.

Time derivatives of the Euler's angles are related to the angular velocities $\omega_1, \omega_2, \omega_3$ by the Euler's kinematic equations:

$$\begin{aligned}
\dot{\phi} &= (\omega_1 \sin\psi + \omega_2 \cos\psi)/\sin\theta, \\
\dot{\theta} &= \omega_1 \cos\psi - \omega_2 \sin\psi, \\
\dot{\psi} &= \omega_3 - \dot{\phi}\cos\theta.
\end{aligned} \tag{19}$$

For the nutation theory of the Earth we can set $\omega_3 = \omega$. Then defining the complex variable D by the relation

$$D = \dot{\theta} + i\dot{\phi}\sin\theta \tag{20}$$

the Euler's kinematic relations reduce to the single complex equation

$$D = u\exp(i\psi)$$

that complements the system of the equations (2)–(5) to a close system of differential equations relatively to the variables ϕ, θ, v_x, v_y. In these equations the rotational angle ψ is a known linear function of time and differs from the Greenwich Sidereal Time by the constant π.

SOS equations for the model without the fluid core reduce to the the single equation:

$$\dot{u}(1 + \sigma e) - ie\omega u(1 - \sigma) = L + i\frac{\sigma}{\omega}\frac{\partial L}{\partial t}. \tag{21}$$

From this equation one can conclude that the Chandler's frequency f^{ch} of free pole oscillations is given by the following expression:

$$f^{ch} = e\omega\frac{1 - \sigma}{1 + \sigma e}, \tag{22}$$

which means that the Euler frequency $e\omega$ of the free oscillations of the rigid Earth is contracted by the factor $1 - \sigma \approx 0.7$ for the elastic Earth.

1.4. Revised dynamical model of the Earth's rotation

In the model developed further, SOS equation (17) will be replaced by the following one:

$$\dot{u}\left(1+\frac{2}{3}e\sigma\right) - ie\omega(1-\sigma)u + \left(\alpha+\frac{2}{3}e\nu\right)(\dot{v}+i\omega v) + iv\sum_{k=1,2}(1-3\zeta_k^2)p_k =$$
$$= L + (\delta+i)\frac{\sigma}{\omega}\frac{\partial L}{\partial t} + L^d + L_c^d \quad (23)$$

in which the normalized dissipative torque L^d consists of the lunar L_1^d and solar L_2^d components caused by the dissipation in the lunar and solar tides, and of the cross interaction torque $L_{1,2}^d$ of these tides:

$$L^d = L_1^d + L_2^d + L_{1,2}^d, \quad (24)$$
$$L_k^d = -4p_k\epsilon_k\sigma\delta\left[\omega\xi_k\zeta_k + i\left(\zeta_k\frac{\partial}{\partial t}\xi_k - \xi_k\frac{\partial}{\partial t}\zeta_k\right)\right] \; (k=1,2), \quad (25)$$
$$L_{1,2}^d = 2\sigma\delta\omega(p_1\epsilon_2 + p_2\epsilon_1)(\xi_2\zeta_1 + \xi_1\zeta_2), \quad (26)$$

while L_c^d includes the terms due to the dissipation in the fluid core:

$$L_c^d = \nu\delta_c\left[\frac{1}{2}pv(3\cos^2\theta - 2\cos\theta - 1) + ie(\alpha-\nu)L\right]. \quad (27)$$

Here p_1, p_2 are parameters of the lunar and solar precession, relatively (see definition (7)), $\epsilon_1 = p_1/e\omega$, $\epsilon_2 = p_2/e\omega$, and $p = p_1 + p_2$.

The revised differential equation for the fluid core has the form

$$\dot{u} + \dot{v} + iv\omega\left[1 + e_c - \frac{\mu e}{\alpha}(1+i\delta_c)\right] = \frac{\nu}{\alpha}\left[L\left(1-\frac{2}{3}e\right) - \frac{i}{\omega}\frac{\partial L}{\partial t}\right] +$$
$$+ i\delta\frac{\nu}{\alpha}\left[L(1-e) + i\left(\frac{2}{\omega}\right)\frac{\partial L}{\partial t}\right] = 0, \quad (28)$$

to be compared with corresponding SOS equation (18).

Equations (23)-(28) refine conventional SOS equations (17), (18) by introducing the two dissipative parameters δ and δ_c in the explicit way. The parameter δ is the effective tidal lag of the Earth as a whole and namely it strongly affects the orbital motion of the Moon being responsible for the evolution of the Earth-Moon system. The parameter δ_c is the phase lag of the tides caused by the differential rotation of the fluid core and as we show further; it plays important part in the Earth's rotation.

Setting the tidal lags δ, δ_c equal to zero one could expect that the two systems of the differential equations become equivalent. However it is easy to see that there is no full equivalence: in equation (23) the factor $1 + 2e\sigma/3$ stands for the factor $1+e\sigma$ in SOS equation (17). The origin of this discrepancy will be clarified in Section 2.7. In brief that is due to the incomplete form of the centrifugal tidal potential used in the conventional model where only the tesseral components of this potential have been accounted for. Omission of the zonal components leads to minor errors of the second order with respect to e and probably does not deteriorates fitting to observations (though the theoretical interpretation of the results may be corrupted).

Equations (23)-(28) show also that any attempts to describe the dissipative effects by a formal consideration of imaginary parts of the Love numbers k_2, k_2^d (or of the compliances κ, γ) are of no physical meaning because the functional dependence of equations (23)-(28) on the phase lags δ, δ_c has another structure, excepting the single term in the right part of equation (23) which is proportional to the derivative $\partial L/\partial t$.

In equations (23)-(28) there are omitted the time-dependent parts of the terms proportional to u. Their impact on the precession and nutation is negligible but they are responsible for tidal variations of the amplitude and phase of the Chandler's wobble on a detectable level. Such effects are beyond the topic of this paper and it is supposed to study them in another place.

In the next section we start with the simplest case of the purely elastic Earth without any fluid core. After that the developed procedure will be generalized to the case of the dissipative Earth still ignoring the fluid core. Then the fluid core are introduced by the Poincare's method, at first ignoring effects of dissipation in the fluid core and then accounting for them. Thus a revised version of equations (17)–(18) will be obtained applying only the basic dynamic principles (the theoretical identity of reciprocity (6) will be taken after Sasao without proves).

2. Deformable Earth without the fluid core

2.1. Euler's differential equations

Rotation of a deformable body will be described by the Euler's equations in the matrix form

$$\frac{d}{dt}(I\overline{\omega}) - (I\overline{\omega}) \times \overline{\omega} = \overline{N}, \tag{29}$$

where I is the matrix of moments of inertia perturbed by tides, the torque $\overline{N}$ is the skew product of $\overline{r}$ and the gradient of the potential W of interaction of the Earth with the outer body of the mass m. This potential may be presented in the form

$$W = W_0 + dW_t + dW_r, \tag{30}$$

in which W_0 is the rigid body component, dW_t is the potential generated by the tides aroused by the perturbing body, and dW_r is that due to the tides by the

Earth's rotation. The matrix I, distorted by action of the potentials dW_t, dW_r, is split in the similar way:

$$I = I_0 + dI_t + dI_r \tag{31}$$

where $I_0 = A$ diag $(1, 1, 1 + e)$ is the unperturbed matrix of inertia and the matrices dI_t, dI_r are induced by the potentials dW_t, dW_r, correspondingly. With the potential W the torque $\overline{N}$ is presented by the following expression:

$$\overline{N} = \overline{r} \times \text{grad } W.$$

This torque may be split in the same way into the three components

$$\overline{N} = \overline{N}_0 + \overline{N}_t + \overline{N}_r,$$

where again N_0 is the rigid body contribution, $\overline{N}_t$, $\overline{N}_r$ are additives caused by the lunar or solar tides, and the tides induced by the Earth's rotation.

As it will be shown further, the conventional theory accounts for the tidal variations dI_t, dI_r of the matrix of inertia but ignores the torques $\overline{N}_t$ and $\overline{N}_r$. In the next two sections there will be derived expressions for the tidally induced potentials dW_t, dW_r, for the tidal variations of the matrix I of inertia, and then for the corresponding torques $\overline{N}_t, \overline{N}_r$. These expressions will be obtained for the general case of the dissipative Earth i.e. accounting the tidal lag δ.

Differential equation (29) may be written in the form

$$\dot{\overline{\omega}} + I_0^{-1} \left[(I_0 \overline{\omega}) \times \overline{\omega} \right] = \overline{L}^{ef}, \tag{32}$$

using the normalized 'effective' torque L^{ef} given by the expression

$$\overline{L}^{ef} = I_0^{-1} \left[\overline{N} - \frac{d}{dt} (dI \overline{\omega}) + (dI \overline{\omega}) \times \overline{\omega} \right], \tag{33}$$

where $dI = dI_t + dI_r$.

2.2. Potential of the tidal interaction with tide arousing body

The elastic Earth is distorted by the combined action of tide arousing bodies and the centrifugal force of the Earth's rotation. Let us consider the first effect. Tidal potential $W(\overline{r}, \overline{r}')$ at the point $\overline{r}$ (geocentric radius-vector) caused by an outer body with the mass m at the point $\overline{r}'$, and evaluated at the point $\overline{r}$ has the form

$$W(\overline{r}, \overline{r}') = mG \frac{r^2}{r'^3} P_2^0 (\cos H),$$

where $\cos H = (\overline{\rho}, \overline{\rho}')$, $\overline{\rho} = \overline{r}/r, \overline{\rho}' = \overline{r}'/r'$, and P_2^0 is Legendre polynomial.

The distorted Earth gives rise to the additional potential dW proportional to the Love number k_2; on the Earth's surface it is given by the expression

$$dW|_{r=R} = k_2 m G \frac{R^2}{r'^3} P_2^0(\cos H).$$

This potential may be continued into the outer space as a harmonic function $dW(\overline{r})$ multiplying its values on the Earth's surface by the factor $(R/r)^3$. Thus

$$dW(\overline{r}) = k_2 m G \frac{R^5}{r^3 r'^3} P_2^0(\cos H). \tag{34}$$

When calculating corresponding contribution to the potential energy of interaction between the tidally distorted Earth and the tide arousing body, the function $dW(\overline{r})$ has to be multiplied by the mass m once more. Because we are going to calculate the tidal torques that act onto the Earth from the side of the perturbing body the sign of the resulting expression must be reversed. As the result the additional tidal potential dW_t of the system of the Earth plus the perturbing body is as it follows:

$$dW_t = -k_2 m^2 G \frac{R^5}{r^3 r'^3} P_2^0(\cos H). \tag{35}$$

In analogous way we can derive expression for the contribution dW_r of the rotational deformation of the Earth to the potential energy of the system. For the centrifugal acceleration $\overline{W}$ of the point at the position $\overline{r}$ within the Earth we have the well known expression:

$$\overline{W} = -\overline{\omega} \times (\overline{\omega} \times \overline{r}) \equiv -\overline{\omega}(\overline{\omega}, \overline{r}) + \overline{r}\omega^2.$$

The acceleration may be presented in the form

$$\overline{W} = \text{grad } W_r + \frac{2}{3}\overline{r}\omega^2, \tag{36}$$

where

$$W_r = -\frac{1}{3}\omega^2 r^2 P_2^0(\cos S), \tag{37}$$

and

$$\cos S = (\overline{\omega}, \overline{\rho})/\omega. \tag{38}$$

The last component in the right part of equation (36) presents a radial acceleration that does not distort the uncompressible Earth and thus does not draw any changes in the Earth's rotation; so it may be disregarded. Again on the Earth's surface the Earth's deformations give rise to the additional potential $k_2 W_r$ which may be continued to the position $\overline{r}$ of the perturbing body of the mass m if one multiplies it by the factor $(R/r)^3$. As it has been noticed above the sign of this potential must be reversed if applied to the Earth's rotation and we have:

$$dW_r = \frac{1}{3} k_2 m \omega^2 \frac{R^5}{r^3} P_2^0(\cos S). \tag{39}$$

In the general case of the dissipative Earth it cannot more be assumed that there is no time delay in action of the tides on the matrix of inertia I and on the perturbing celestial bodies. Thus we have to calculate the angular velocity $\overline{\omega}$ that enters the equation for dW_r at the delayed moment $t' = t - \tau$ where the time delay τ characterizes rheology of the Earth. The value of τ may be estimated from analysis of astronomical observations of different kinds but at present the most reliable estimates have been derived from Lunar Laser Ranging data.

In this paper the prime symbol $'$ at some variable marks the tidally delayed time argument of this variable. In particular, in the dissipative case the angle S in expression (39) for the potential dW_r is to be given by the expression

$$\cos S = (\overline{\omega}', \overline{\rho})/\omega. \tag{40}$$

In virtue of the differential equations of rotation of the rigid Earth the value $|\dot{\omega}_3|$ is much smaller than $|\dot{\omega}_1|, |\dot{\omega}_2|$ and thus in relation (40) we can set $\omega_3' = \omega_3, \;\; \omega' = \omega$.

To obtain the tidal torques that act upon the rotating Earth due to the tidally induced potential $dW' = dW_r' + dW_t'$ we must calculate the skew product $\overline{r} \times \text{grad}\; dW'$ in which the gradient vector is evaluated respectively to the variable $\overline{r}$ at the point $\overline{r} = \overline{r}'$. One can see that for the elastic Earth the torque caused by dW_t vanishes being proportional to $\overline{r} \times \overline{r}$. However that holds true only for the purely elastic Earth when the tidal dissipation in the Earth's body is negligible but not in the general case of the anelastic Earth (see Section 2.2). As dW_t, dW_r will be used only to calculate the tidal torques that perturb the Earth's rotation, the spherically symmetric terms may be disregarded and then the expressions for dW_t', dW_r' reduce to the following form:

$$dW_t' \;\; = \;\; -\frac{3}{2} k_2 G m^2 \frac{R^5}{r^5 r'^5} (\overline{r}, \overline{r}')^2, \tag{41}$$

$$dW_r' \;\; = \;\; \frac{1}{2} k_2 m \left(\frac{R}{r}\right)^5 (\overline{\omega}', \overline{r})^2. \tag{42}$$

Potential dW'_t describes interaction of the perturbing body with the tides aroused by this body (the Moon or Sun). Apart from this effect it is necessary to take into consideration the potential of interaction of the tides aroused by the Moon with the Sun, and the analogous interaction of the solar tides with the Moon. If m_1, m_2 are masses of the Moon and Sun the corresponding potentials $W_t^{1,2}, W_t^{2,1}$ are given by the identical expressions

$$W_t^{1,2} = W_t^{2,1} = -k_2 m_1 m_2 G \frac{R^5}{r_1^3 r_2^3} P_2^0(\cos H), \tag{43}$$

where $\cos H = (\overline{r}_1, \overline{r}_2)/r_1 r_2$ is geocentric angle between the Moon and Sun, $\overline{r}_1, \overline{r}_2$ are geocentric vectors to the Sun and Moon.

The two corresponding torques being calculated as the skew products $\overline{r} \times$ grad dW come into opposite directions and so cancel each other. But that is not more true for the anelastic Earth (see Section (2.4)).

Let us consider the potential (30) and the corresponding torque for the case of the purely elastic Earth. The rigid body potential W_0 of interaction of the Earth with the perturbing body may be written in the form:

$$W_0 = Gmm_E \frac{R^2}{r^3} J_2^{(0)} P_2^0(\cos S). \tag{44}$$

The zero upper index at the coefficient $J_2^{(0)}$ means that its value has to be cleared off from the contribution due to the elastic response of the Earth on its rotation. The observed value J_2 of the standard models of the geopotential is connected with the parameter $J_2^{(0)}$ by the identity

$$Gmm_E \frac{R^2}{r^3} J_2 P_2^0(\cos S) \equiv W_0 + dW_r = Gmm_E \frac{R^2}{r^3} \left(J_2^{(0)} + \frac{k_2 R^3 \omega^2}{3Gm_E} \right) P_2^0(\cos S),$$

that involves the relation

$$J_2 = J_2^{(0)} + \frac{k_2 R^3 \omega^2}{3Gm_E} \equiv J_2^{(0)} + \sigma J_2,$$

where σ is defined by the first of expressions (11).

One can see that σJ_2 presents the permanent part of the tidal perturbations of J_2 caused by the Earth's rotation. Thus the unperturbed value $J_2^{(0)}$ that enters the differential equations of the Earth's rotation may be obtained by the following expression

$$J_2^{(0)} = J_2(1 - \sigma). \tag{45}$$

The coefficient J_2 of the geopotential as provided by results of Earth's satellite tracking and given in the IERS standards includes the permanent part of the pole tides caused by the Earth's rotation, while in equations of the Earth's rotation (32), (33) the unperturbed value $J_2^{(0)}$ has to be used.

2.3. Tidal variations of the moments of inertia

Having calculated the potential of tidal interaction of the Earth with tide arousing bodies, we can derive the corresponding tidal contributions to the kinetic energy of the Earth's rotation. They depend on variations of the tensor of inertia I given by relation (31) in which $I_0/A = \text{diag}\ (1, 1, 1 + e)$, e is the dynamical flattening of the Earth, the matrix $dI_t = \{c_{ik}^t\}$ presents the variations of I due to the tides aroused by the perturbing body of the mass m, while $dI_r = \{c_{ij}^r\}$ presents those due to the pole tides aroused by the Earth's rotation.

The components of matrices dI_t and dI_r may be found from the condition that the potential induced by the Earth's deformation when expressed in terms of the moments of inertia is equal to dW_t and dW_r relatively.

Thus if c_{ik}^t are the components of the matrix dI_t then the potential at the point $\overline{r}$ out the Earth, being expressed in terms of moments of inertia c_{ij}^t must be equalized to the right part of equation (35):

$$\begin{aligned} dW_t &= \frac{G}{r^5} \times \\ &[\frac{1}{2}x^2(c_{22}^t + c_{33}^t - 2c_{11}^t) + \frac{1}{2}y^2(c_{33}^t + c_{11}^t - 2c_{22}^t) + \frac{1}{2}z^2(c_{11}^t + c_{22}^t - 2c_{33}^t) \\ &-3xyc_{12}^t - 3xzc_{13}^t - 3yzc_{23}^t]. \end{aligned} \tag{46}$$

It is known (see Moritz & Mueller, 1987, [12]) that for elastic body the trace of the perturbed matrix of inertia is equal to zero. In our case it means that $c_{11}^t + c_{22}^t + c_{22}^t = 0$. Making use of this equality, and comparing equations (35) with (46) we obtain the following expressions for the matrix coefficients c_{ij}^t

$$\frac{c_{ii}^t}{Ap_t'} = \frac{1}{3} - {\rho_i'}^2, \tag{47}$$

$$\frac{c_{ij}^t}{Ap_t'} = -\rho_i' \rho_j' (i \neq j), \tag{48}$$

where $\rho_1' = x_1'/r', \rho_2' = x_2'/r', \rho_3' = x_3'/r'$ are coordinates of the unit vector $\overline{\rho}$ to the perturbing body, and the undimensional variable $p_t(r)$ is given by the expression

$$p_t(r) = k_2 \frac{R^5}{r^3 A} = k_2 \left(\frac{R}{r}\right)^3 \frac{m}{m_E g}. \tag{49}$$

In the dissipative case the parameter p_t in relation (47) depends on the argument r'. If there were no dissipation then coordinates (ρ_1, ρ_2, ρ_3) and $(\rho_1', \rho_2', \rho_3')$ would coincide; thus the prime symbol $'$ might be omitted.

It is easily can be verified that the parameter p_t may be presented in the form

$$p_t = 2\sigma \frac{p}{\omega}, \tag{50}$$

where the constant σ is defined by the first of relations (11), and p is the parameter of precession from the perturbing body of the mass m given by relation (7).

Further when necessary the variable p will be presented as p_1 (for the Moon) or p_2 (for the Sun).

Calculating in the analogous way the corrections $dI_r = c^r_{ij}$ to the tensor of inertia caused by the rotational deformations we obtain:

$$\frac{c^r_{ii}}{Ap_r} = -\frac{1}{3} + \frac{{\omega'_i}^2}{\omega^2}, \tag{51}$$

$$\frac{c^r_{ij}}{Ap_r} = \frac{\omega'_i \omega'_j}{\omega^2}(i \neq j), \tag{52}$$

where the undimensional constant p_r is given as follows:

$$p_r = k_2 \frac{R^5 \omega^2}{3GA} = k_2 \frac{R^3 \omega^2}{3Gm_E g} = \sigma \frac{J_2}{g} = \sigma e = \frac{k_2}{k_s} e. \tag{53}$$

It is useful to take in mind that the rotational perturbations of the matrix of inertia greatly exceed those from the luni-solar tides as $p_t << p_r$:

$$\frac{p_t}{p_r} = 2\frac{p}{\omega e} \approx 10^{-5}.$$

This ratio enters equations of the Earth's rotation in a number of places and for convenience we define the small non-dimensional constant ϵ_k by the expression

$$\epsilon_k = \frac{p_k}{\omega e}, \tag{54}$$

where p_1, p_2 are lunar and solar precessional parameters. Note that ϵ_k and p_k are constant only when eccentricity of the perturbing body is ignored.

2.4. The elastic Earth

Rotation of the elastic body will be described in the matrix form by equations (32), (33) where the time dependent matrix dI of the tidal corrections to moments of inertia is the sum of the two components presented by expressions (47), (48), and (51), (52), relatively. The effective torque $\overline{L}^{ef}$ at the right parts is calculated assuming $\rho = \rho'$, $\overline{\omega}' = \overline{\omega}'$. For the purely elastic Earth we have to set $\overline{r}' = \overline{r}$ in equation (41) after calculating the gradients. The torques from the luni-solar tides vanish as they are proportional to the skew product $\overline{r} \times \overline{r} = 0$. For the normalized torque $\overline{L}_r$ caused by the rotational tidal potential (42) we have

$$\overline{L}_r = \overline{r} \times \text{grad } \frac{dW_r}{A} = p_t(\overline{\rho} \times \overline{\omega})(\overline{\rho}, \overline{\omega}).$$

Now let us calculate the rigid body torque $\overline{N}_0 = \overline{r} \times \text{grad} W_0$ writing W_0 in the form:

$$W_0 = Gmm_E \frac{R^2}{r^3} J_2 P_2^0(\cos S). \tag{55}$$

Let $\overline{\omega}_0 = (0, 0, \omega)$ be the unperturbed vector of the angular velocity. Calculating the skew product it may be easily verified that the following expression holds true for the normalized rigid body torque $\overline{L}_0 = \overline{N}_0/A$:

$$\overline{L}_0 = 2\frac{p^0}{\omega}(\overline{\rho} \times \overline{\omega}_0)(\overline{\rho}, \overline{\omega}_0), \tag{56}$$

where p^0 is the parameter of precession calculated with the value of the dynamical flattening e_0 obtained by subtracting the permanent tide caused by the Earth's rotation:

$$p^0 = \frac{3}{2} \frac{Gm}{\omega r^3} e_0. \tag{57}$$

The relation between e and e_0 follows from equality (45):

$$e_0 = e - p_r = e(1 - \sigma).$$

It is important to pay attention that the theoretical value e given in the models of the Earth's interior (for instance in the PREM model) corresponds to the Earth flattened by its rotation due to the Earth's elasticity and so $e > e_0$.

Thus for the resulting normalized torque $\overline{L}$ calculated as the skew product

$$\overline{L} = \overline{r} \times \text{grad } (W_0 + dW_r)\frac{1}{A},$$

the following equality holds true:

$$\overline{L} = \overline{L}_0 + \sigma \frac{2p^0}{\omega}(\overline{\rho} \times \overline{\omega})(\overline{\rho}, \overline{\omega}), \tag{58}$$

in which the parameter p_t has been replaced by its expression from equation (50).

To complete the calculation of all tidal terms in the equation of rotation we must derive the terms generated by the tidal deformations of the matrix of inertia. Due to relations (51), (52) we have

$$\frac{dI_r}{A}\overline{\omega} = \frac{2}{3}e\sigma\overline{\omega} \tag{59}$$

and thus the rotational deformations dI_r of the tensor of inertia do not contribute to the skew product term $I\overline{\omega} \times \overline{\omega}$ in equations (32), (33):

$$\frac{1}{A}\left(dI_r\overline{\omega}\right) \times \overline{\omega} = (\overline{\omega} \times \overline{\omega})e\sigma\omega^2 = 0. \tag{60}$$

It is noteworthy to mention a deficiency of the commonly used way to derive the analytical expression of the Chandler's frequency (see for instance the monograph (Moritz & Mueller, 1987, [12]). In this approach it is assumed that only the tesseral part of the tidally induced potential (42) affects the nutations taking this potential is in the reduced form:

$$dW_r = k_2 m \left(\frac{R}{r}\right)^5 (\omega_1 x_1 + \omega_1 x_2)x_3. \tag{61}$$

As a result instead of equality (59) that draws the identity (60), one obtains the relation

$$\frac{dI_r}{A}\overline{\omega} = e\sigma \begin{pmatrix} \omega_1 \\ \omega_2 \\ 0 \end{pmatrix}, \tag{62}$$

and thus (neglecting the second order terms respectively to ω_1, ω_2)

$$\frac{1}{A}\left(dI_r\overline{\omega}\right) \times \overline{\omega} = e\sigma\omega \begin{pmatrix} -\omega_2 \\ \omega_1 \\ 0 \end{pmatrix}, \tag{63}$$

In a lucky way expression (63) when inserted to equations of the Earth's rotation yields correct expression (22) for the Chandler's frequency because no accounting for the permanent rotational tide in the dynamical flattening e is applied. However the usage of the abridged relation (61) instead of its full form (42) involves not only some minor errors in the conventional differential equations of nutation (due to the differences of expressions (59) and (62)) but prevents proper modeling of the dissipative effects in terms of the effective phase lag δ, which is the main drawback of the conventional model.

For the perturbation dI_t of the inertia matrix caused by the tides from the outer body we have

$$\frac{dI_t}{A}\overline{\omega} = p_t\left(\frac{1}{3}\overline{\omega} - \overline{\rho}(\overline{\rho},\overline{\omega})\right) = 2\sigma\frac{p}{\omega}\left(\frac{1}{3}\overline{\omega} - \overline{\rho}(\overline{\rho},\overline{\omega})\right), \tag{64}$$

with the corresponding component in the skew product:

$$dI_t\overline{\omega}\times\overline{\omega} = Ap_t\left(\frac{1}{3}\overline{\omega} - \overline{\rho}(\overline{\rho},\overline{\omega})\right)\times\overline{\omega} = -Ap_t(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega}).$$

Neglecting the terms of the second order, equations (32), (33) may be rewritten in the following form:

$$\frac{d\overline{\omega}}{dt} + e_0(\overline{\omega}\times\overline{\omega}_0) + p_t(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega}) + \frac{1}{A}\frac{d}{dt}(dI_r\overline{\omega} + dI_t\overline{\omega}) = \overline{L}, \tag{65}$$

where the right hand is given by relations (56)-(58). In accordance with equation (58) the third term at the left side cancels the analogous term at the right hand; replacing e_0 by $e(1-\sigma)$ after some simplifications we obtain

$$\frac{d\overline{\omega}}{dt} + e(1-\sigma)(\overline{\omega}\times\overline{\omega}_0) + \frac{1}{A}\frac{d}{dt}(dI_r\overline{\omega} + dI_t\overline{\omega}) = 2\frac{p^0}{\omega_0}(\overline{\rho}\times\overline{\omega}_0)(\overline{\rho},\overline{\omega}_0). \tag{66}$$

Making use of equation (51) for the coefficients c^r_{ij} of matrix dI_r and equality (53) for the constant p_r we have the following expression for the time derivative of $dI_r\overline{\omega}$ that enters equation (66):

$$\frac{d}{dt}\left(\frac{dI_r}{A}\overline{\omega}\right) = \frac{2}{3}p_r\dot{\overline{\omega}} \equiv \frac{2}{3}e\sigma\dot{\overline{\omega}} \tag{67}$$

Applying relation (1) we obtain for the time derivative of dI_t:

$$\frac{d}{dt}\left(\frac{dI_t}{A}\overline{\omega}\right) = \frac{\partial}{\partial t}p_t\left[\frac{1}{3}\overline{\omega} - \overline{\rho}(\overline{\rho},\overline{\omega})\right] + p_t(\overline{\omega}\times\overline{\rho})(\overline{\omega},\overline{\rho}). \tag{68}$$

Then equation (66) reduces to the form:

$$\left(1 + \frac{2}{3}e\sigma\dot{\overline{\omega}}\right)\frac{d\overline{\omega}}{dt} + e(1-\sigma)(\overline{\omega}\times\overline{\omega}_0) = \overline{L} + \overline{L}^e, \tag{69}$$

where $\overline{L}^e$ is given by expression

$$\overline{L}^e = \frac{2\sigma}{\omega}\frac{\partial}{\partial t}p\left[\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right]. \tag{70}$$

Expression (58) for $\overline{L}$ may be rewritten in the form

$$\overline{L} = \frac{2p}{\omega}(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega}) + \left[\frac{2p^0}{\omega_0}(\overline{\rho}\times\overline{\omega}_0)(\overline{\rho},\overline{\omega}_0) - \frac{2p^0}{\omega}(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega})\right], \tag{71}$$

where $p = p^0(1+\sigma)$ includes the component from the permanent pole tide.

The last two terms at the right part cancel each other at $\overline{\omega} = \overline{\omega}_0$, so they are proportional to ω_1, ω_2 and may be neglected in the differential equations of precession and nutation. Thus the expression for $\overline{L}$ (in which the rigid body torque are united with the torque of the similar structure generated by the permanent luni-solar tide) will be written in the form

$$\overline{L} = \frac{2p}{\omega}(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega})$$

and for brevity will be referred further as the rigid body torque.

Though perturbations in the nutations produced by the neglected terms in equation (71) indeed are ignorable, these terms give rise to measurable effects in the form of the tidal variations of the amplitude and phase of the free pole motion. Such effects will be studied in another place.

Now let us write down equations (69) for the components ω_1, ω_2 in terms of the complex variable $u = \omega_1 + i\omega_2$ to compare them with SOS equations (21) reduced to the case of the purely elastic Earth without any fluid core. From the above considerations the following expressions are valid for the components L_1, L_2 of the normalized torque $\overline{L}$ (including the permanent tide):

$$\begin{aligned} L_1 &= 2p\omega\rho_2\rho_3, \\ L_2 &= -2p\omega\rho_1\rho_3. \end{aligned}$$

Introducing the complex coordinates ξ, ζ instead of the coordinates ρ_1, ρ_2, ρ_3 of the unit vector to the perturbing body

$$\xi = \rho_1 + i\rho_2, \zeta = \rho_3, \tag{72}$$

the complex presentation of the normalized rigid body torque $L = L_1 + iL_2$ may be presented by the simple expression

$$L = -2ip\omega\xi\zeta. \tag{73}$$

In analogous way the complex presentation $L^e = L_1^e + iL_2^e$ of the vector $\overline{L}^e$ in equation (70) has the form

$$L^e = i\frac{\sigma}{\omega}\frac{\partial L}{\partial t}$$

and as the result the differential equations for the variable u may be written in the form:

$$\dot{u}\left[1 + \frac{2}{3}e\sigma(1+\epsilon)\right] - iuf^{ch} = L + i\frac{\sigma}{\omega}\frac{\partial L}{\partial t}, \tag{74}$$

where the time dependent parameter f^{ch} denotes the Chandler's frequency

$$f^{ch} = e\omega\left[1 - \sigma - \epsilon\sigma\frac{2\dot{p}}{3p\omega}\right]. \tag{75}$$

This equation has to be compared with the conventional equation (21). One can see that the conventional equation does not include the small ϵ term generated by the luni-solar tides but only the terms induced by the tides from the Earth's rotation. Moreover, even neglecting the ϵ terms the two formulations differ in the scaling factors (the coefficients at $\dot{u}$), the ratio of which is about 1.001. As the scaling factor is one of the parameters under estimation in fitting to observations by the method of transfer function, this deficiency of the adopted model does not influence on the quality of the fitting but only on physical interpretation of the results.

The standard formulation ignores the term $\dot{p}$ that describes a weak time dependence (due to the luni-solar tides) of parameters of the free oscillations of the Earth's pole. Just as the terms omitted by us in relation (71) the omission does not affect nutations; when studying the Chandler's wobble the resulting effects are also negligible.

The advantage of the developed more rigorous approach is that it may be easily generalized to the case of the anelastic Earth with non-zero tidal lag. A detail consideration of these effects (which are not negligible but missed in the conventional SOS equations) is given in the next section.

2.5. The anelastic Earth

2.5.1. The tidal phase delay

Rigorous equations of rotation of the anelastic Earth depend on some variables which must be calculated for the time argument delayed by the value τ. Direct numerical integration of these equations seems to be the most adequate method to obtain the solution of the highest accuracy. And indeed in this way the numerical theory of rotation of the deformable Moon has been constructed in the well-known DE ephemerides and then successfully applied to analysis of LLR observations. Such approach seems very promising for the Earth's rotation as well, though it has never before been realized. In this method some technical problems arise: firstly because the differential equations are not of a standard type but are the equations with the retarded argument, and secondly because the dissipation leads to dumping of free oscillations and thus the backward in time integration becomes unstable. The experience with the theory of the Moon's rotation has demonstrated that these problems may be overcome. Numerical integration of equations of rotation of the Moon proved to be the only method with capacity to provide lunar ephemerides adequate in accuracy to contemporary LLR observations. Situation with the Earth's rotation seems very similar. For the numerical integration it is preferable to have right parts of differential equations in the simplest form while for analytical methods of the perturbation theory as well as for a qualitative analysis (which is the aim of this study) they must be transformed in such a way to get rid of the retarded time argument. That may be fulfilled if the variables with the retarded argument are approximated by linear terms of the Taylor series relatively to the time delay τ:

$$\overline{\rho}' = \overline{\rho}(t-\tau) \approx \overline{\rho} - \tau \frac{d\overline{r}}{dt}, \qquad (76)$$

$$\overline{\omega}' = \overline{\omega}(t-\tau) \approx \overline{\omega} - \tau \frac{d\overline{\omega}}{dt}. \qquad (77)$$

Commonly it is assumed that the physically correct approach is to consider as the constant value not the tidal time delay τ but rather the tidal phase lag δ defined by the relation

$$\delta = \omega\tau.$$

Discerning these two assumptions is important only for problems of the tidal evolution. In the theory of nutation even processing the most accurate contemporary observations, the angular velocity ω may be considered constant and the two formulations are equivalent.

The tidal variations of the matrix I of the moments of inertia have to be calculated for the tidally delayed time argument and thus the splitting of I given by relation (31) must be corrected to the form

$$I = I_0 + dI_t' + dI_r',$$

with the following expressions for the components, instead of relations (59), (64):

$$\frac{1}{A} dI_r' \overline{\omega} = -\frac{p_r}{3}\overline{\omega} + p_r \overline{\omega}'(\overline{\omega}', \overline{\omega}) \frac{1}{\omega^2},$$

$$\frac{1}{A} dI_t' \overline{\omega} = \frac{p_t'}{3}\overline{\omega} - p_t' \overline{\rho}'(\overline{\rho}', \overline{\omega}),$$

in which $p'_t = p_t(\rho')$.

As $|\dot\omega_3|$ is much smaller than $|\dot\omega_1|, |\dot\omega_2|$, the expression for $dI'_r\overline{\omega}$ may be taken in the form:

$$\frac{1}{A} dI'_r\overline{\omega} = p_r\left(-\frac{1}{3}\overline{\omega} + \overline{\omega}'\right). \tag{78}$$

Then for the skew product $(dI'_r\overline{\omega}) \times \overline{\omega}$ we obtain

$$\frac{1}{A}(dI'_r\overline{\omega}) \times \overline{\omega} = p_r(\overline{\omega}' \dot{\times} \omega). \tag{79}$$

For the luni-solar component dI'_t instead of relations (64) and (68) we have

$$\frac{dI'_t}{A}\overline{\omega} = p'_t\left[\frac{\overline{\omega}}{3} - \overline{\rho}'(\overline{\rho}', \overline{\omega})\right]. \tag{80}$$

$$\frac{d}{dt}\left(\frac{dI'_t}{A}\right)\overline{\omega} = \frac{\partial}{\partial t}p'_t\left[\frac{1}{3}\overline{\omega} - \overline{\rho}'(\overline{\rho}', \overline{\omega})\right] + p'_t(\overline{\omega} \times \overline{\rho}')(\overline{\omega}, \overline{\rho}'). \tag{81}$$

In the case of the anelastic Earth it is also necessary to use the more rigorous expression for the torque $\overline{N}$ at the right part of expression (33):

$$\overline{N} = \overline{r} \times \text{grad } (W_0 + dW'_t + dW'_r)$$

in which the tidal time delay in the constituents dW'_r, dW'_t is accounted. Instead of expression (58) for the normalized torque $\overline{L} = \overline{N}/A$ we have now:

$$\overline{L} = L_0 + p_t\left[-3\frac{mG}{r'^3}(\overline{\rho} \times \overline{\rho}') + (\overline{\rho} \times \overline{\omega}')(\overline{\rho}, \overline{\omega}')\right] + p_r(\overline{\omega}' \times \overline{\omega}), \tag{82}$$

where $\overline{L}_0$ is given by relation (56).

Generalizing the differential equations (66) of the Earth's rotation to the anelastic case they may be presented in the form

$$\frac{d\overline{\omega}}{dt} + e(1-\sigma)(\overline{\omega} \times \overline{\omega}_0) = \overline{L}^{ef},$$

in which the normalized effective torque $\overline{L}^{ef}$ is given (making use of relations (78)–(82)) by the following expression:

$$\overline{L}^{ef} = \overline{L}_0 + p_t\left[-3\frac{mG}{r'^3}(\overline{\rho} \times \overline{\rho}') + (\overline{\rho} \times \overline{\omega}')(\overline{\rho}, \overline{\omega}')\right] - p'_t(\overline{\rho}' \times \overline{\omega})(\overline{\rho}', \overline{\omega}) +$$
$$+ p_r(\overline{\omega}' \times \overline{\omega}) + p_r\left(\frac{1}{3}\dot{\overline{\omega}} - \dot{\overline{\omega}}'\right) + \frac{d}{dt}p'_t\left[\overline{\rho}'(\overline{\rho}', \overline{\omega}) - \frac{1}{3}\overline{\omega}\right]. \tag{83}$$

The vector $\overline{L}^{ef}$ may be evaluated retaining the terms of the first degree respectively to the time delay τ and written in the form:

$$\overline{L}^{ef} = \overline{L} + \overline{L}^{e} + \overline{L}^{d}, \tag{84}$$

where the components $\overline{L}$, $\overline{L}^{e}$ are presented by expressions (58), (70), relatively, while the dissipative component L^d consists of the terms which vanish when $\tau = 0$:

$$\begin{aligned} \overline{L}^{d} &= -3p_t \frac{mG}{r'^3}(\overline{\rho} \times \overline{\rho}') + \\ &+ [p_t(\overline{\rho} \times \overline{\omega}')(\overline{\rho}, \overline{\omega}') - p_t'(\overline{\rho}' \times \overline{\omega})(\overline{\rho}', \overline{\omega})(\overline{\omega}', \overline{\omega})] + \\ &+ \tau \frac{d^2}{dt^2} p_t' \left[\overline{\rho}'(\overline{\rho}', \overline{\omega}) - \frac{1}{3}\overline{\omega}\right] + p_r(\overline{\omega}' \times \overline{\omega}) - p_r \dot{\overline{\omega}}'. \end{aligned} \tag{85}$$

The second and third terms (placed in braces) cancel at $\tau = 0$ and in linear in τ approximation they may be presented as follows:

$$p_t(\overline{\rho} \times \overline{\omega}')(\overline{\rho}, \overline{\omega}') - p_t'(\overline{\rho}' \times \overline{\omega})(\overline{\rho}', \overline{\omega})(\overline{\omega}', \overline{\omega}) = -\tau \frac{d}{dt} p_t(\overline{\rho} \times \overline{\omega}')(\overline{\rho}, \overline{\omega}'). \tag{86}$$

Applying identity (1) to the second order derivative at the right part of equation (85) we have

$$\frac{d^2}{dt^2} p_t \left[\overline{\rho}(\overline{\rho}, \overline{\omega}) - \frac{1}{3}\overline{\omega}\right] = \frac{d}{dt}\frac{\partial}{\partial t} p_t \left[\overline{\rho}(\overline{\rho}, \overline{\omega}) - \frac{1}{3}\overline{\omega}\right] - p_t(\overline{\omega} \times \overline{\rho})(\overline{\rho}, \overline{\omega}), \tag{87}$$

and substituting expressions (86) and (87) to equation (85) the last term in (87) cancels the term (86). Then equation (85) reduces to the form:

$$\begin{aligned} \overline{L}^{d} &= -3p_t \frac{mG}{r'^3}(\overline{\rho} \times \overline{\rho}') - p_r(\dot{\overline{\omega}}' - \overline{\omega}' \times \overline{\omega}) + \\ &+ \tau \frac{d}{dt}\frac{\partial}{\partial t} \left[p_t \left(\overline{\rho}(\overline{\rho}, \overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right]. \end{aligned} \tag{88}$$

The terms at the right part of this relation will be transformed in the following way:

1. The skew product $\overline{\rho} \times \overline{\rho}'$ is written making use of relation (76):

$$\begin{aligned} (\overline{\rho} \times \overline{\rho}') &= -\tau \overline{\rho} \times \frac{d}{dt}\overline{\rho} = -\tau \rho \times \left(\frac{\partial}{\partial t}\overline{\rho} - \overline{\omega} \times \overline{\rho}\right) - \tau \left[\overline{\rho} \times \frac{\partial}{\partial t}\overline{\rho} - \overline{\rho} \times (\overline{\omega} \times \overline{\rho})\right] \\ &= \tau \left[\overline{\omega} - \overline{\rho}(\overline{\rho}, \overline{\omega}) - \overline{\rho} \times \frac{\partial}{\partial t}\overline{\rho}\right], \end{aligned} \tag{89}$$

(the identity $\overline{A} \times (\overline{B} \times \overline{C}) = \overline{B}(\overline{A}, \overline{A}) - \overline{C}(\overline{A}, \overline{B})$ being used).

2. The last term in equation (88) after applying identity (1) becomes

$$\tau \frac{d}{dt}\frac{\partial}{\partial t}\left[p_t\left(\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right] = \tau\frac{\partial^2}{\partial t^2}\left[p_t\left(\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right] - \\ -\tau\frac{\partial}{\partial t}p_t(\overline{\omega}\times\overline{\rho})(\omega,\overline{\rho}). \tag{90}$$

3. The combination $\dot{\overline{\omega}}' - \overline{\omega}' \times \overline{\omega}$ vanishes at $\tau = 0$ and thus with sufficient accuracy it may be evaluated in virtue of the simplified differential equations written in the form:

$$\dot{\overline{\omega}} = \overline{L} - e(1-\sigma)(\overline{\omega} \times \overline{\omega}_0).$$

.

Then ignoring the higher orders we obtain

$$\overline{\omega}' \times \overline{\omega} - \dot{\overline{\omega}}' = -\tau\dot{\overline{\omega}} \times \overline{\omega} + \tau\ddot{\overline{\omega}} = \\ -\tau\left(\dot{\overline{L}} - \overline{L}\times\overline{\omega}\right) + \tau e(1-\sigma)\left[\overline{\omega}\times(\overline{\omega}\times\overline{\omega}_0)\right],$$

and after further transformations

$$\overline{\omega}' \times \overline{\omega} - \dot{\overline{\omega}}' = -\tau\frac{\partial \overline{L}}{\partial t} + e(1-\sigma)\omega\left[\dot{\overline{\omega}} \times \overline{\omega}_0 + \omega(\overline{\omega} - \overline{\omega}_0)\right]$$

(identity (1) being applied once more to calculate $\dot{\overline{L}}$). The first term at the right part has the same structure that the last term at the right part of equality (90) but in the equations of rotation it is multiplied by the small factor $p_r = \sigma e$; so it may be ignored. The other terms in the right part vanish when $\omega_1 = \omega_2 = 0$ and their perturbations in the forced nutation are negligible as well. However they must be accounted in equations of the free pole motion if one studies tidal variations of its parameters; in particular they describe the tidal damping of the Chandler's wobble.

4. The coefficient mG/r^3 at the right part of equation (85) may be presented in terms of the parameter ϵ defined by expression (54) in the following way:

$$3\frac{mG}{r^3} = \omega^2\frac{2}{e}\left(\frac{p}{\omega}\right) = 2\epsilon\omega^2, \tag{91}$$

that follows from the defining equality (7) for the precessional parameter p and from relation (54).

Substituting relations (89)–(91) to equation (88), and using expressions (50), (53) for the tidal parameters p_t and p_r we obtain the following expression for the effective dissipative torque (85):

$$\overline{L}^d = -4p\epsilon\sigma\delta\left(\overline{\omega} - \overline{\rho}(\overline{\rho},\overline{\omega}) - \overline{\rho}\times\frac{\partial}{\partial t}\overline{\rho}\right) - 2\left(\frac{\sigma\delta}{\omega^2}\right)\frac{\partial}{\partial t}p(\overline{\rho}\times\overline{\omega})(\overline{\rho},\overline{\omega}) + \\ +2\left(\frac{\sigma\delta}{\omega^2}\right)\frac{\partial^2}{\partial^2 t}\left[p\left(\overline{\rho}(\overline{\rho},\overline{\omega}) - \frac{1}{3}\overline{\omega}\right)\right]. \quad (92)$$

2.5.2. Dissipative cross effect of the luni-solar tides

The right parts of equations (70), (92) have been written for a single perturbing body of the mass m; for all the perturbing bodies the corresponding expressions must be summarized (detectable contributions being due to action of the Moon and Sun). So in a more rigorous form of presentation the effective torques $\overline{L}^e, \overline{L}^d$ and the variables $\overline{\rho}, p, \epsilon$ must be supplied by the index 1 for the lunar and 2 for the solar terms. There is also a dissipative torque $\overline{L}^d_{1,2}$ caused by the tidal cross-interaction of the Moon and Sun, which must be added after the summation. Thus the full form of the normalized effective torque $\overline{L}^{ef}$ is as follows

$$\overline{L}^{ef} = \sum_{k=,1,2}\left(\overline{L}_k + \overline{L}^e_k + \overline{L}^d_k\right) + \overline{L}^d_{1,2},$$

where $\overline{L}_k$ are the rigid body lunar and solar torques, $\overline{L}^e_k$, $\overline{L}^d_k$ are the elastic and dissipative contributions; these terms are given by expressions (70) and (92) supplied by the corresponding index $k = 1, 2$.

Let us derive expression for the term $L^d_{1,2}$ in a more detail, denoting the masses of Moon and the Sun as m_1, m_2, the potential of the solar tides acting upon the Moon as $W_t^{1,2}$, and that of the lunar tides acting upon the Sun as $W_t^{2,1}$ (see equation (43) for the case of the purely elastic Earth). Geocentric vectors to the Sun and Moon will be noted as $\overline{r}_1$ and $\overline{r}_2$. Because we are going to calculate the skew products of these vectors with gradients of the corresponding potentials, the spherically symmetric parts of the potentials may be omitted, the rest having the following form:

$$W_t^{1,2} = -\frac{3}{2}k_2 m_1 m_2 G\frac{R^5}{{r'_1}^3 r_2^3}(\overline{\rho}'_1, \overline{\rho}_2)^2, \quad (93)$$

$$W_t^{2,1} = -\frac{3}{2}k_2 m_1 m_2 G\frac{R^5}{r_1^3 {r'_2}^3}(\overline{\rho}_1, \overline{\rho}'_2)^2, \quad (94)$$

where $\overline{\rho}_1 = \overline{\rho}_1/\rho_1, \overline{\rho}_2 = \overline{\rho}_2/\rho_2$ are unit vectors to the Moon and Sun.

The normalized sum $\overline{L}^d_{1,2}$ of the both torques is

$$\overline{L}^d_{1,2} = \overline{r}_2 \times \text{grad}_{\overline{r}_2}\left(\frac{W_t^{1,2}}{A}\right) + \overline{r}_1 \times \text{grad}_{\overline{r}_1}\left(\frac{W_t^{2,1}}{A}\right).$$

In the elastic case $\overline{r}_1 = \overline{r}'_1$, $\overline{r}_2 = \overline{r}'_2$ and these two terms cancel each other. With the accuracy of the first order respectively to τ we have

$$\overline{L}^d_{1,2} = -3\tau m_1 m_2 \frac{R^5 G}{r_1^3 r_2^3 A}\left(\overline{\rho}_2 \times \frac{d\overline{\rho}_1}{dt} + \overline{\rho}_1 \times \frac{d\overline{\rho}_2}{dt}\right)(\overline{\rho}_1, \overline{\rho}_2). \tag{95}$$

It can be easily verified that the factor at the right hand of this equality may be presented in terms of solar p_1 and lunar p_2 precessional parameters:

$$3\delta k_2 m_1 m_2 \frac{R^5 G}{r_1^3 r_2^3 A\omega} = 4 p_1 p_2 \frac{\sigma\delta}{e}.$$

Applying relation (1) to the sum of the skew products at the right part of equation (95) we have:

$$\begin{aligned}
&\overline{\rho}_2 \times \frac{d\overline{\rho}_1}{dt} + \overline{\rho}_1 \times \frac{d\overline{\rho}_2}{dt} = \\
&\overline{\rho}_2 \times \frac{\partial\overline{\rho}_1}{\partial t} + \overline{\rho}_1 \times \frac{\partial\overline{\rho}_2}{\partial t} - \left[\overline{\rho}_2 \times (\overline{\omega} \times \overline{\rho}_1) + \overline{\rho}_1 \times (\overline{\omega} \times \overline{\rho}_2)\right] = \\
&\frac{\partial}{\partial t}\left(\overline{\rho}_2 \times \overline{\rho}_1 + \overline{\rho}_1 \times \overline{\rho}_2\right) - \overline{\rho}_2(\overline{\rho}_1, \overline{\omega}) - \overline{\rho}_1(\overline{\rho}_2, \overline{\omega}) + 2\overline{\omega}(\overline{\rho}_1 \overline{\rho}_2) = \\
&-\overline{\rho}_2(\overline{\rho}_1, \overline{\omega}) - \overline{\rho}_1(\overline{\rho}_2, \overline{\omega}) + 2\overline{\omega}(\overline{\rho}_1 \overline{\rho}_2),
\end{aligned}$$

(here the identity $\overline{A} \times (\overline{B} \times \overline{C}) = \overline{B}(\overline{A}, \overline{C}) - \overline{C}(\overline{A}, \overline{B})$ has been used once more). Thus for the normalized torque $\overline{L}^d_{1,2}$ we obtain the expression

$$\overline{L}^d_{1,2} = 4\pi_{1,2}\sigma\delta\left[\overline{\rho}_2(\overline{\rho}_1, \overline{\omega}) + \overline{\rho}_1(\overline{\rho}_2, \overline{\omega}) - 2\overline{\omega}(\overline{\rho}_1 \overline{\rho}_2)\right](\overline{\rho}_1, \overline{\rho}_2), \tag{96}$$

in which

$$\pi_{1,2} = 4\frac{p_1 p_2}{e\omega}.$$

The cross-factor $\pi_{1,2}$ may be written in the symmetric form:

$$\pi_{1,2} = 2(p_1\epsilon_2 + p_2\epsilon_1),$$

where parameters ϵ_k are defined by expression (54) for solar p_1 and lunar p_2 parameters of precession.

The vectorial presentation of the effective torques describes the time behavior both of the axis of rotation $\overline{\omega}/\omega$, and the axial angular velocity ω (and thus of the Universal Time). The equations for ω_3 are independent of those for ω_1, ω_2 and written separately may be presented in a more simple analytical form.

2.6. Tidal variations of the angular velocity ω

With sufficient accuracy the relation $\dot{\overline{\omega}} \approx \dot{\overline{\omega}}_3$ holds true and thus the right part of the differential equation for $\overline{\omega}$ is the projection of the effective torque $\overline{L}^{ef}$ onto the axis ρ_3. The rigid body torques does not contribute to $\dot{\omega}$ and we have:

$$\frac{\dot{\omega}}{\omega} = \sum_{k=1,2} \left(V_k^e + V_k^d\right) + V_{1,2}^d, \tag{97}$$

where V_k^e, V_k^d and $V_{1,2}^d$ are the projections of the effective torques $\overline{L}_k^e$, L_k^d and $\overline{N}_{1,2}$, divided by ω, onto this axis.

In order to write down the solar, lunar and cross-interaction terms in a uniform way, in this subsection (and only there) we make use of another notation for the coordinates of the unit vectors $\overline{\rho}_1, \overline{\rho}_2$ to the Moon and Sun

$$\overline{\rho}_k = (\xi_k,\ \eta_k,\ \zeta_k),\ \ k = 1, 2.$$

In these notations the term V_k^e due to elasticity has the form

$$V_k^e = 2\sigma \frac{\partial}{\partial t} \left[p_k \left(\zeta_k^2 - \frac{1}{3} \right) \right]. \tag{98}$$

Note that all functions met in this subsection do not depend on the rotational angle ψ and thus there is no difference between the partial and full time derivatives. The skew product $\overline{\rho} \times \overline{\omega}$ in the right part of expression (92) is orthogonal to the axis x_3 and so does not contribute to equation (97); thus we have the following expressions for V_k^d $(k = 1, 2)$:

$$\begin{aligned} V_k^d = -4 p_k \epsilon_k \delta \left[1 - \zeta_k^2 + \frac{1}{\omega} \left(\eta_k \frac{\partial}{\partial t} \xi_k - \xi_k \frac{\partial}{\partial t} \eta_k \right) \right] + \\ +2\sigma \frac{\delta}{\omega} \frac{\partial^2}{\partial t^2} \left[p_k \left(\zeta_k^2 - \frac{1}{3} \right) \right], \end{aligned} \tag{99}$$

and for $V_{1,2}^d$:

$$V_{1,2}^d \quad = \quad 4\delta\sigma (p_1 \epsilon_2 + p_2 \epsilon_1) \left(\zeta_1 \zeta_2 \cos H - \cos^2 H \right), \tag{100}$$

where

$$\cos H = (\overline{\rho}_1, \overline{\rho}_2),$$

H being the geocentric angle between the Moon and Sun.

The tidal variations of the angular velocity ω caused by elasticity may be presented after integration of equation (98) in a close form; they are studied in a number of works (see for instance (Yoder et al, 1981, [18]). The dissipative perturbations presented in the form (99) were deduced in (Krasinsky, 2000, [5]) and applied to study of the tidal evolution of the Earth-Moon system (Krasinsky, 2002, [6]). To our knowledge the dissipative cross term (100) is obtained here at the first time. It affects the evolution of the Earth-Moon system but has never been accounted in studies of the evolution.

2.7. Equations for precession and nutation

Projections of the both parts of the vectorial differential equation (32) to the axes ρ_1 and ρ_2, being combined with Euler's kinematic equations (19) make close system of differential equations for the Euler's angles of precession ϕ and nutation θ. In expressions (70), (92) for $\overline{L}^e_k$, $\overline{L}^d_k$ $(k=1,2)$, and (96) for $\overline{L}^d_{1,2}$ we ignore the terms which vanish with ω_1, ω_2 because they affect noticeably only the free polar motion. The second partial derivative $\partial^2/\partial t^2$ also will be ignored as the corresponding perturbations in nutations do not exceed one μas.

Equations of precession and nutation have the simplest form written in the complex variables $u = \omega_1 + i\omega_1$ for the angular velocity, and

$$\xi = \rho_1 + i\rho_2, \;\; \zeta = \rho_3 \tag{101}$$

for the coordinates of the tide arousing bodies. In these variables various complex expressions that enter the differential equations may be presented in the following form:

1. The combination $L_1^{1,2} + iL_2^{1,2}$ of the coordinates of the vector $\overline{L}^d_{1,2}$ given by equality (96) becomes:

$$L_1^{1,2} + iL_2^{1,2} = 2(p_1\epsilon_2 + p_2\epsilon_1)\sigma\delta\left[\xi_2\zeta_1 + \xi_1\zeta_2 - 2\frac{u}{\omega}(\overline{\rho}_1, \overline{\rho}_2)\right](\overline{\rho}_1, \overline{\rho}_2). \tag{102}$$

2. The vector $\overline{n} = \overline{\rho} \times \partial\overline{\rho}/\partial t$ in equation (92) is directed along the normal to the orbital plane. For its coordinates $n_1,\ n_2$ the following relations may be easily verified:

$$\begin{aligned} n_1 &= \rho_2\dot{\rho}_3 - \rho_3\dot{\rho}_2, \\ n_2 &= \rho_1\dot{\rho}_3 - \rho_3\dot{\rho}_1 \end{aligned}$$

and as the result in the complex variables ξ, ζ we have the relation

$$n_1 + in_2 = i(\zeta\dot{\xi} - \xi\dot{\zeta}). \tag{103}$$

Now the differential equation of precession and nutation in the complex variable u may be written in the following form:

$$\dot{u}(1 + \frac{2}{3}e\sigma) - ie\omega(1-\sigma)u = L^{ef}, \tag{104}$$

where the normalized effective torque L^{ef} is the sum of the main component $L^{(m)}$ that includes the complex presentation L of the rigid body torque and its partial time derivatives $\partial L/\partial t$, and the small additional component $L^{(ad)}$ that absorbs all other terms:

$$L^{ef} = L^{(m)} + L^{(ad)}. \tag{105}$$

In this structure the main part $L^{(m)}$ of the effective torque presents the perturbing terms for which the analytical solution may be presented in the form of a differential operator applied to the nutations calculated in the rigid body approximation. In the adopted terminology this part of the full solution may be obtained by the method of the transfer function. The additional term $L^{(ad)}$ gives rise to small but not negligible perturbations in the nutation, precession and obliquity rate which cannot be presented by any transfer function and need more detail treating.

Making use of relations (92), (96), and (102), (103) we have the following expression for $L^{(m)}$ which includes the rigid body torque and the luni-solar terms from both the elasticity and dissipation:

$$L^{(m)} = L + (\delta + i)\frac{\sigma}{\omega}\frac{\partial L}{\partial t}. \tag{106}$$

Here the rigid body torque $L = L_1 + L_2$ is the sum of the lunar L_1 and solar L_2 components calculated with the help of expression (73) in which either $p = p_1, m = m_1$ or $p = p_2, m = m_2$, and the dynamical flattening e includes the permanent rotational tide (and so is consistent with the IERS value).

The additional effective torque

$$L^d = L_1^d + L_2^d + L_{1,2}^d \tag{107}$$

consists of the small lunar and solar components caused by the dissipation in the lunar and solar tides, and the dissipative cross term of the interaction of these tides:

$$L_k^{(ad)} = -4p_k\epsilon_k\sigma\delta\left[\omega\xi_k\zeta_k + i\left(\zeta_k\frac{\partial}{\partial t}\xi_k - \xi_k\frac{\partial}{\partial t}\zeta_k\right)\right], \tag{108}$$

$$L_{1,2}^d = 2\sigma\delta\omega(p_1\epsilon_2 + p_2\epsilon_1)(\xi_2\zeta_1 + \xi_1\zeta_2). \tag{109}$$

3. Deformable Earth with a fluid core

3.1. General equations by Poincare

General differential equations of the Earth's rotation in the variables $\overline{\omega}$, $\overline{v}$ will be derived here by the Poincare's method after (Moritz & Mueller, 1987, [12]). The kinetic energy T of the Earth with a fluid core may be written in the form:

$$T = \frac{1}{2}\left[(I\overline{\omega}, \omega) + (I^c\overline{v}, \overline{v}) + 2(I^c\overline{\omega}, \overline{v})\right],$$

where I, I^c are matrices of inertia of the Earth and its fluid core. Then the equations of rotation will be as follows:

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \overline{\omega}} + \overline{\omega} \times \frac{\partial T}{\partial \overline{\omega}} &= \overline{N}, \\ \frac{d}{dt}\frac{\partial T}{\partial \overline{v}} - \overline{v} \times \frac{\partial T}{\partial \overline{v}} &= 0, \end{aligned}$$

where $\overline{N}$ are the luni-solar torques.

In a more detail (neglecting the terms of the second order respectively to $\overline{v} = (v_1, v_2, 0)$) we have to write:

$$\frac{d}{dt}(I\overline{\omega} + I^c\overline{v}) + \overline{\omega} \times I\overline{\omega} + \overline{\omega} \times I^c\overline{v} = \overline{N}, \tag{110}$$

$$\frac{d}{dt}(I^c\overline{\omega} + I^c\overline{v}) - \overline{v} \times I^c\overline{\omega} = 0. \tag{111}$$

Accounting for the tidal variations of the matrices of inertia I, I^c and the corresponding contributions to the torques $\overline{N}$ these equations generalize those given by Poincare for the case of a rigid body with a fluid core. The variations of the matrices I, I^c are produced by combined action of the luni-solar tides and the tides aroused by centrifugal forces. Further the consideration of these effects, presented in the previous sections, is duly modified and as the result more complete differential equations of rotation of the deformable Earth with the viscous fluid core will be derived.

3.2. Tidal potential due to the centrifugal acceleration

In the previous sections there are given expressions for the time dependent variations of the matrix of inertia I due to the tides from outer bodies (the component I_t) and from the Earth's rotation (the component I_r). Now we have to deduce expressions for the tidally perturbed part of I caused by the differential rotation of the fluid core, and that of I^c produced by perturbing tidal potentials. For simplicity we start from the idealized case of the nondissipative mantle and core.

First of all we have to obtain the correction to the geopotential induced by distortion of the mantle and fluid core by the centrifugal force. For that equality (37) which presents the centrifugal acceleration must be generalized to account for the differential rotation of the fluid core taking into consideration the angular velocity $\overline{v}$ of this rotation. The velocity of any point $\overline{r}$ within the core may be presented as $(\overline{\omega} + \overline{v}) \times r$, while within the mantle it is equal to $\overline{\omega} \times r$; thus the centrifugal acceleration $\overline{W}$ at any point within the Earth is given by the expressions:

$$\begin{aligned}\overline{W} &= -\overline{\omega}\times(\overline{\omega}\times\overline{r}) = -\overline{\omega}(\overline{r},\overline{\omega}) + \overline{r}\omega^2 \text{ in the mantle,}\\ \overline{W} &= -(\overline{\omega}+\overline{v})\times[(\overline{\omega}+\overline{v})\times\overline{r}] = -(\overline{\omega}+\overline{v})\left[(\overline{r},\overline{\omega})+(\overline{r},\overline{v})\right] + \overline{r}|\overline{\omega}+\overline{v}|^2\\ &\quad \text{in the core.}\end{aligned}$$

The last (radial) terms in the both expressions do not deform the uncompressible Earth and may be disregarded. Then $\overline{W}$ can be presented as $\overline{W} = \text{grad } W_r$ where the potential W_r (in which the terms of the second order are ignored) is as follows:

$$W_r = -\frac{1}{2}(\overline{r},\overline{\omega})^2 \text{ in the mantle,} \quad (112)$$

$$W_r = -\frac{1}{2}(\overline{r},\overline{\omega})^2 - (v_1x_1 + v_2x_2)x_3\omega \text{ in the core.} \quad (113)$$

Supplementing the right part by the term $\frac{3}{2}r^2\omega^2$ (that does not influence on distribution of density of the uncompressible Earth) we obtain:

$$\begin{aligned}W_r &= -\frac{1}{3}\omega^2r^2P_2^0(\cos S) \text{ in the mantle,}\\ W_r &= -\frac{1}{3}\omega^2r^2P_2^0(\cos S) - (v_1x_1+v_2x_2)x_3\omega \text{ in the core.}\end{aligned}$$

In accordance with the general theory of Love numbers this potential deforms the Earth's interior and as the result the deformations induce the additional potential dW_r on the Earth's surface ($\overline{r} = \overline{R}$):

$$dW_r|(\overline{r}=\overline{R}) = -\frac{k_2}{3}\omega^2r^2P_2^0(\cos S) - k_2^d(v_1x_1+v_2x_2)x_3\omega,$$

k_2, k_2^d being the static and dynamic Love numbers.

Multiplied by the factor $(R/r)^3$ this potential may be continued to the outer space. In order to describe the action of the perturbing body on the Earth it is necessary to reverse the sign of dW_r and to multiply dW_r by the mass m of this body. The first term with the factor k_2 has been already considered in Section 2.2 for the case of the deformable Earth without the fluid core and thus expression (39) is still valid. The second term is responsible for the effects of the fluid core and will be referred as dW_f:

$$dW_f = k_2^d m\left(\frac{R}{r}\right)^5(v_1\rho_1+v_2\rho_2)\rho_3\omega, \quad (114)$$

where $\overline{\rho} = (\rho_1, \rho_2, \rho_3)$ is the unit vector to the perturbing outer body of the mass m.

Making use of the relation

$$k_2^d m \left(\frac{R}{r}\right)^5 \frac{1}{A} = 2\nu \frac{p}{\omega},$$

where p is the parameter of precession from the perturbing body of the mass m we can write the expression for dW_f in the form

$$\frac{dW_f}{A} = 2\nu p(v_1\rho_1 + v_2\rho_2)\rho_3. \tag{115}$$

This expression will be used for calculation of the perturbations in the same way as that has been done with expressions (41), (42) when calculating the tidal torques induced by the luni-solar tides, and by the tides due to the Earth's rotation, as well as the tidal variations of the matrix of inertia I.

3.3. The tidal torque induced by the fluid core

Now we have to derive an expression for the additional torque $\overline{N}^f$ caused by interaction of the potential dW_f induced by the rotating fluid core (equation (115)) with perturbing bodies (the Moon and Sun). This torque significantly contributes to the Earth's rotation but is not accounted in the adopted nutational theory.

For the normalized torque $\overline{L}^f$ we have

$$\overline{L}^f = \overline{r} \times \text{grad}\, \frac{dW_f}{A}$$

and due to expression (114) for dW_f

$$\text{grad}\, \frac{dW_f}{A} = \frac{2\nu p'}{\omega} \begin{pmatrix} v_1'\rho_3' \\ v_2'\rho_3' \\ v_1'\rho_1' + v_2'\rho_2' \end{pmatrix},$$

where again the symbol $'$ marks the variables with arguments delayed by the tidal dissipation.

Then the torque $\overline{L}^f$ may be presented as follows:

$$\overline{L}^f = 2\nu p' \begin{pmatrix} v_1'\rho_1'\rho_2 + v_2'(\rho_2\rho_2' - \rho_3\rho_3') \\ -v_1'(\rho_1\rho_1' - \rho_3\rho_3') - v_2'\rho_1\rho_2' \\ (v_1'\rho_2 - \rho_1 v_2')\rho_3' \end{pmatrix}.$$

In this equality it is sufficient to account the tidal lag only in the fast variables (which depend on the rotational angle ψ), thus we set $p' = p$, $\rho_3' = \rho_3$. Other variables with the delayed arguments will be evaluated in the linear approximation respectively to the phase lag δ_c. (Note that there is no reason to assume $\delta_c = \delta$). The torque $\overline{L}^f$ may be expressed as the sum of the elastic response $\overline{L}^{f,e}$ and the dissipative component $\overline{L}^{f,d}$ that vanishes with the tidal lag:

$$\overline{L}^f = \overline{L}^{f,e} + \overline{L}^{f,d},$$

where

$$\overline{L}^{f,e} = 2\nu p \begin{pmatrix} v_1\rho_1\rho_2 + v_2(\rho_2^2 - \rho_3^2) \\ -v_1(\rho_1^2 - \rho_3^2) - v_2\rho_1\rho_2 \\ (v_1\rho_2 - \rho_1 v_2)\rho_3 \end{pmatrix},$$

and

$$\overline{L}^{f,d} = 2\nu p \begin{pmatrix} \rho_2(v_1'\rho_1' - v_1\rho_1) + \rho_2(v_2'\rho_2' - v_2\rho_2) - \rho_3^2(v_2' - v_2) \\ -\rho_1(v_2'\rho_2' - v_2\rho_2) - \rho_1(v_1'\rho_1' - v_1\rho_1) + \rho_3^2(v_1' - v_1) \\ (v_1 - v_1')\rho_2\rho_3 - (v_2 - v_2')\rho_1\rho_3 \end{pmatrix}. \tag{116}$$

In the expression for $\overline{L}^{f,e}$ the product $\rho_1\rho_2$ generates nothing but very small semi-diurnal terms in nutations, and may be ignored. The combinations ρ_1^2, ρ_2^2, ρ_3^2 must be averaged relatively to the rotational angle ψ because the time dependent parts give rise only to sub-diurnal perturbations of the Euler's angles. In Appendix A for these averaged values the following expressions are obtained:

$$\langle\rho_1^2\rangle = \langle\rho_1^2\rangle = \frac{1}{2}(1 - \rho_3^2). \tag{117}$$

Then for the complex presentation $L^{f,e} = L_1^{f,e} + iL_2^{f,e}$ of the components $L_1^{f,e}$, $L_2^{f,e}$ we have:

$$L^{f,e} = -iv(1 - 3\zeta^2)p\nu$$

or (to indicate explicitly the order of smallness of this value)

$$L^{f,e} = -iv\omega e(1 - 3\zeta^2)\nu\epsilon, \tag{118}$$

where ϵ is the small undimensional parameter defined by relation (54).

Expression for the dissipative part $L^{f,d}$ will be derived in subsection 3.6 with other contributions due to the dissipation in the fluid core.

3.4. The Earth as a whole: differential equation for u

In order to account the fluid core we must modify expression (31) when splitting the matrix of inertia I of the Earth onto the sum of the unperturbed matrix I_0, and the contributions dI_t, dI_r of the tides induced by outer bodies and the Earth's rotation. Namely, the additional term $dI_f = c^f_{ij}$ has to be added to this expression to account for the tides caused by the differential rotation of the fluid core:

$$I = I_0 + dI_t + dI_r + dI_f. \tag{119}$$

Analytical expressions for the matrices dI_t, dI_r have been derived in previous sections. To calculate the coefficients c^f_{ij} of dI_f we can apply relations (51), (52) replacing ω_1, ω_2 by v_1, v_2 and assuming $v_3 = 0$. Then we obtain

$$c^f_{i3} = c^f_{3i} = A\nu e \frac{v'_i}{\omega} \text{ for } i = 1, 2 \tag{120}$$

and $c^f_{ij} = 0$ for other combinations of the indices. As before the prime symbol $'$ means that the time argument is delayed due to the tidal dissipation.

The matrix dI_f enters the left part of the vectorial differential equation (110) as a vector $\overline{s}^f$ of the following form:

$$\overline{s}^f = \frac{d}{dt}(dI_f \overline{\omega}) + \overline{\omega} \times (dI_f \overline{\omega}).$$

Applying relations (120) we obtain

$$\frac{\overline{s}^f}{A} = e\nu(\dot{\overline{v}}' + \overline{\omega} \times \overline{v}'),$$

or in the complex presentation $s^f = s^f_1 + i s^f_2$:

$$\frac{s^f}{A} = e\nu(\dot{v}' + i\omega v'). \tag{121}$$

The perturbed matrix of inertia I^c of the fluid core may be given in the form analogous to the presentation (119) of the matrix I:

$$I^c = I^c_0 + dI^c_t + dI^c_r + dI^c_f, \tag{122}$$

where I^c_0 is the unperturbed matrix, dI^c_t is the contribution from the luni-solar tides, dI^c_r, dI^c_f are those due to rotation of the mantle and the differential rotation of the fluid core. The component dI^c_f may be ignored because it is proportional to v and being multiplied by v in equations (110) gives rise to the terms of the second order.

It is naturally to assume that the reciprocity principle (6) by Sasao is valid not only for the non-diagonal terms of the perturbed matrices of inertia but also for the diagonal ones. Then the matrices dI^c_t, dI^c_r and dI_t, dI_r will be connected by the relations

$$dI^c_t = \frac{k^d_2}{k_2} dI_t, \;\; dI^c_r = \frac{k^d_2}{k_2} dI_r$$

and the vectors $dI^c_t\overline{\omega}$ and $dI^c_r\overline{\omega}$ that enter equation (110) may be obtained with the help of equations (59), (64), replacing the factor σ by the factor ν.

It is also necessary to obtain the vector $\overline{c} = I^c\overline{v}$ that enters equations (110), (111). We present this vector as

$$\overline{c} = I^c\overline{v} = \overline{c}^0 + \overline{c}^t + \overline{c}^r + \overline{c}^f \tag{123}$$

where the splitting corresponds to that in the right part of equation (122). For our aims we need the projections c_1, c_2 on the axes ρ_1 and ρ_2 only in the form of the complex combination $c = c_1 + ic_2$; the corresponding components of c are denoted as c^0, c^t, c^r, c^f. The vector $\overline{c}$ enters equation (110) in combination with the vectorial product as the vector $\overline{c}_\omega = \dot{\overline{c}} + \overline{\omega} \times \overline{c}$. In analogous way we split $\overline{c}_\omega$ to the sum

$$\overline{c}_\omega \equiv \dot{\overline{c}} + \overline{\omega} \times \overline{c} = \overline{c}^0_\omega + \overline{c}^t_\omega + \overline{c}^r_\omega + \overline{c}^f_\omega$$

using the notation $c^0_\omega, c^t_\omega, c^r_\omega, c^f_\omega$ for the complex presentation of coordinates of these vectors. Analytical expressions for these complex variables are as it follows:

1. Unperturbed components c^0_ω.

As $I^c_0 = A_c \mathrm{diag}(1, 1, 1 + e_c)$ we have

$$\begin{aligned} \frac{c^0}{A_c} &= v, \\ \frac{c^0_\omega}{A_c} &= \dot{v} + iv. \end{aligned} \tag{124}$$

2. The terms c^r_ω.

Applying expressions (51)–(53) to compute $\overline{c}^r = dI^c_r\overline{v}$ we obtain:

$$\begin{aligned} \frac{c^r}{A} &= -\frac{e\nu}{3} v, \\ \frac{c^r_\omega}{A} &= -\frac{e\nu}{3}(\dot{v} + iv\omega). \end{aligned} \tag{125}$$

3. The terms c^t_ω.

To compute $\overline{c}^t = dI^c_t \overline{v}$ we apply relation (64) replacing σ, $\overline{\omega}$ by ν, $\overline{v}$ and obtain (in the complex presentation)

$$\frac{c^t}{A} = \frac{2p\nu}{\omega}\left[\frac{1}{3}v - (\rho_1 + i\rho_2)(v_1\rho_1 + v_2\rho_2)\right].$$

Here we retain only the mean values $\langle\rho_1^2\rangle = \langle\rho_2^2\rangle$ making use of expressions (117). In this approximation

$$\frac{c^t}{A} = -v\frac{p\nu}{6\omega}(1 - 3\zeta^2) = -\frac{1}{6}v\nu e\epsilon(1 - 3\zeta^2).$$

With the same accuracy the eccentricity of the perturbing body may be neglected and the parameter ϵ considered constant; then for c^t_ω we obtain

$$\frac{c^t_\omega}{A} = \frac{1}{6}\nu e\epsilon(1 - 3\zeta^2)(\dot{v} + i\omega v).$$

In the rigorous formulation the coordinate ζ of the time arousing body must be calculated for the tidally delayed time argument; however we ignore this effect in such a small term. Moreover one can see that c^t_ω is ϵ times less than c^r_ω and has the same structure; so it also will be disregarded.

Now divide the both parts of equation (110) by the moment of inertia A substituting expressions (121), (124), (125) for the constituents of matrices of inertia I and I^c, presented in the form of relations (119) and (122), and making use of expression (118) for the the additional torque $L^{f,e}$ induced by the fluid core. The unperturbed constituent (124) enters the equation with the factor α. Repeating the considerations of the previous sections for the terms that do not depend on the angular velocity v of the fluid core we obtain the differential equation which generalizes equation (104) to the case of the deformable Earth with the dissipative fluid core and the dissipative mantle:

$$\dot{u}\left(1 + \frac{2}{3}e\sigma\right) - ie\omega(1-\sigma)u + \left(\alpha - \frac{1}{3}e\nu\right)(\dot{v} + i\omega v) + e\nu(\dot{v}' + i\omega v')$$
$$+iv\sum_{k=1,2}(1 - 3\zeta_k^2)p_k = L + (\delta + i)\frac{\sigma}{\omega}\frac{\partial L}{\partial t} + L^d + L^{f,d}, \qquad (126)$$

where the torque L^d is given by by expressions (107)–(109), and $L^{f,d}$ is the complex presentation of the first two components of the vector $\overline{L}^{f,d}$ given by equation (116).

That is a preliminary form of the equation to be used, because it still depends on the variable v', ρ_1', ρ_2' with the delayed time argument. In Section 3.6

the conclusive form of this equation will be given expressing the delayed variable v' in terms of v and the tidal lag δ_c.

Setting $\delta_c = 0$ (which means that $v' = v$) equation (126) may be compared with corresponding equation (17) for the SOS model. Apart of the explained above difference in the scaling factor (the coefficient at $\dot{u}$) one can see that the e-dependent part of the coefficient at the combination $\dot{v} + i\omega v$ is $\frac{2}{3}$ of that in the conventional theory. The origin of this difference is clear: that is again due to the usage of the incomplete form (61) of the Earth's tidal potential induced by the centrifugal acceleration instead of rigorous expressions (38), (39).

3.5. The core: differential equation for the variable v

When computing in equation (111) the vector $\overline{s}^0 == \overline{v} \times I^c \overline{\omega}$ we can set the matrix I^c equal to its unperturbed value I_0^c. Then in the complex presentation $s^0 = s_1^0 + is_2^0$ we obtain

$$\frac{s^0}{A_c} = -iv\omega(1 + e_c), \tag{127}$$

where $e_c = (C_c - A_c)/C_c$ is the dynamical flattening of the core.

In equations (111) there enter the vectors $\overline{s}_r, \overline{s}_t, \overline{s}_f$ defined by the relations

$$\overline{s}_r = \frac{d}{dt}(dI_r^c \overline{\omega}), \ \ \overline{s}_t = \frac{d}{dt}(dI_t^c \overline{\omega}), \ \ \overline{s}_f = \frac{d}{dt}(dI_f^c \overline{\omega}), \tag{128}$$

expressions for which are derived in the following way:

1. To obtain $dI_r^c \overline{\omega}$, $dI_r^c \overline{\omega}$ we can use relations (78) (81) replacing σ by ν in the expressions (49), (53) for p_t, p_r. Then in the complex presentation we have:

$$\frac{s_r}{A} = \nu e \left(-\frac{1}{3}\dot{u} + \dot{u}' \right), \tag{129}$$

$$\frac{s_t}{A} = \frac{2\nu}{\omega} \frac{\partial}{\partial t} \left(\frac{1}{3} p' u - p' \xi' \zeta' \omega \right) + 2\nu p' i \xi' \zeta' \omega. \tag{130}$$

Neglecting in expression (130) the component $p'u$ (as it affects mainly the polar motion) the term s_t may be expressed through the rigid body torque L given by equality (73):

$$\frac{s_t}{A} = -\nu \left(L' + \frac{i}{\omega} \frac{\partial L'}{\partial t} \right). \tag{131}$$

2. The matrix dI_f^c presents perturbations of the matrix of inertia I^c of the fluid core induced by the differential rotation v. To compute the vector $\overline{s}_f = dI_f^c \overline{\omega}$ we can apply equation (120) for the matrix dI_f replacing ν by μ. In the complex presentation we find

$$\frac{s_f}{A} = \mu e \dot{v}'. \tag{132}$$

Combining this expression with relations (124), (125) we can present equations (111) in the form:

$$\dot{u} + \dot{v} + iv\omega(1+e_c) + \left(\frac{e\nu}{\alpha}\right)\left(\dot{u}' - \frac{\dot{u}}{3}\right) - \frac{\nu}{\alpha}\left(L' + \frac{i}{\omega}\frac{\partial L'}{\partial t}\right) + \frac{\mu}{\alpha} e\dot{v}' = 0. \tag{133}$$

Setting $\delta = \delta_c = 0$, the reduced equation

$$\dot{u} + \dot{v} + iv\omega(1+e_c) + \frac{2}{3}\left(\frac{e\nu}{\alpha}\right)\dot{u} - \frac{\nu}{\alpha}\left(L + \frac{i}{\omega}\frac{\partial L}{\partial t}\right) + \frac{\mu}{\alpha} e\dot{v} = 0.$$

may be compared with that of the SOS model which in our notations has the form:

$$\dot{u} + \dot{v} + iv\omega(1+e_c) + \frac{e\nu}{\alpha}\dot{u} - \frac{\nu}{\alpha}\left(L + \frac{i}{\omega}\frac{\partial L}{\partial t}\right) + \frac{\mu}{\alpha} e\dot{v} = 0.$$

The coefficient at the fourth term in our formulation is 2/3 of that in the SOS model. Again this is the result of the incomplete form of the centrifugal potential in the conventional model as it has been shown in Section 2.1.

3.6. Dissipation in the fluid core

First of all let us simplify expression (116) for the dissipative part $\overline{L}^{f,d}$ of the torque caused by the fluid core, making use of the approximation

$$v' = v - \tau_c \dot{v} = v - \delta_c \frac{\dot{v}}{\omega}. \tag{134}$$

When evaluating $\overline{L}^{f,d}$ the arriving combinations ρ_3^2, $\dot{\rho}_1\rho_2$, $\dot{\rho}_2\rho_1, \dot{\rho}_1\rho_1, \dot{\rho}_2\rho_2$ will be replaced by their averaged values. As it is shown in Appendix A the following equalities are valid:

$$\langle \dot{\rho}_1 \rho_2 \rangle = -\langle \dot{\rho}_2 \rho_1 \rangle = -\frac{1}{2}\omega \cos\theta, \tag{135}$$

$$\langle \dot{\rho}_3^2 \rangle = \frac{1}{2}\sin^2\theta, \tag{136}$$

$$\langle \dot{\rho}_1 \rho_1 \rangle = \langle \dot{\rho}_2 \rho_2 \rangle = 0, \tag{137}$$

where averaging is produced relatively to both the rotational angle ψ and the orbital longitude of the perturbing body.

Evaluating expression (116) for $\overline{L}^{f,d}$ we can replace ρ_3^2 by its mean value $\langle\rho_3^2\rangle$, and $\dot{v}_1, \dot{v}_2$ by $\omega v_2, -\omega v_1$, relatively. Then we obtain

$$\overline{L}^{f,d} = \frac{1}{2}\nu p \delta_c (3\cos^2\theta - 2\cos\theta - 1)\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix},$$

or in the complex presentation

$$L^{f,d} = \frac{1}{2}\nu p v \delta_c (3\cos^2\theta - 2\cos\theta - 1). \tag{138}$$

Rigorously speaking this relation have been derived for solar and lunar components separately and p means either p_1 or p_2. However after replacing the squares ζ_k^2 by their averaged values (which do not depend on the perturbing bodies in the approximation used) expression (138) is a correct presentation of the sum of these two components, if p means the parameter of the luni-solar precession.

When calculating the variables with the delayed time argument in differential equations (126), (133) we use the phase lags δ and δ_c for L' and v', relatively. First of all let us evaluate $\dot{v}' + i\omega v'$ in equation (126). The arising derivatives $\dot{v}, \ddot{v}$ in the small tidal terms may be calculated in virtue of differential equations (126), (133) in which $v' = v$. For that it is sufficient to use the following strongly simplified version of these equations in the form:

$$\dot{u} = L,, \tag{139}$$

$$\dot{u} + \dot{v} + iv\omega = \frac{\nu}{\alpha}L. \tag{140}$$

Excluding $\dot{u}$ from the second equation we have:

$$\dot{v} + i\omega v = -\left(1 - \frac{\nu}{\alpha}\right)L. \tag{141}$$

Calculating the time derivative of the both parts, the term $\dot{L}$ arises which again can be evaluated in virtue of the differential equations. We have to apply identity (1) written in the complex presentation for the vector $\overline{L}$ in the following way:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} - iL. \tag{142}$$

Then ignoring the second order derivative $\partial^2 L/\partial t^2$ we obtain:

$$\dot{L} = \frac{\partial L}{\partial t} - i\omega L$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial t}\right) = -i\omega\frac{\partial L}{\partial t}.$$

Calculating the combination $\ddot{v} + i\omega\dot{v}$ the partial derivative $\partial L/\partial t$ may be ignored (but it must be retained when calculating the variables that depend on the phase lag δ and are not multiplied by the small factor e). In this approximation we obtain:

$$\ddot{v} + i\omega\dot{v} = i\omega\left(1 - \frac{\nu}{\alpha}\right)L. \tag{143}$$

Thus

$$\dot{v}' + i\omega v' = \dot{v} + i\dot{v} - \tau(\ddot{v} + i\omega\dot{v}) = \dot{v} + i\omega v - i\delta_c\left(1 - \frac{\nu}{\alpha}\right)L.$$

Substituting this relation to equation (126) it becomes

$$\dot{u}\left(1 + \frac{2}{3}e\sigma\right) - ie\omega(1-\sigma)u + \left(\alpha + \frac{2}{3}e\nu\right)(\dot{v} + i\omega v) + ivK =$$
$$= L + (\delta + i)\frac{\sigma}{\omega}\frac{\partial L}{\partial t} + L^d + L^d_c, \tag{144}$$

where

$$K = \sum_{k=1,2}(1 - 3\zeta_k^2)p_k, \tag{145}$$

L^d being given by expressions (107)–(109), and L^d_c presenting all the terms that depend on the phase lag δ_c of the fluid core:

$$L^d_c = \nu\delta_c\left[\frac{1}{2}pv(3\cos^2\theta - 2\cos\theta - 1) + ie(\alpha - \nu)L\right]. \tag{146}$$

Now let us turn to differential equation (133) and transform it approximating the variables u', L' and v' by linear functions of δ, and δ_c respectively, and applying relations (76), (77), and (134). The arising time derivatives will be evaluated in virtue of the simplified differential equations (139), (140). Then we have

$$\dot{u}' = \dot{u} - \tau\ddot{u} = \dot{u} - \frac{\delta}{\omega}i\dot{L} = \dot{u} + \delta\left(iL - \frac{\partial L}{\partial t}\right).$$

Here the partial derivative $\partial L/\partial t$ may be ignored because in equation (133) the term u' is multiplied by the small factor e. Moreover in this equation all terms multiplied by the flattening e (including the non-dissipative components) also may be evaluated in virtue of the simplified differential equations; thus with the sufficient accuracy we obtain:

$$\dot{u}' - \frac{1}{3}\dot{u} = \frac{2}{3}L + i\delta L. \tag{147}$$

Making use of identity (142) the luni-solar tidal terms in equation (133), reduce to the form

$$\left(L' + \frac{i}{\omega}\frac{\partial L'}{\partial t}\right) = \left(L + \frac{i}{\omega}\frac{\partial L}{\partial t}\right) - \frac{\delta}{\omega}\left(\dot{L} + \frac{i}{\omega}\frac{\partial \dot{L}}{\partial t}\right) =$$

$$= \left(L + \frac{i}{\omega}\frac{\partial L}{\partial t}\right) + i\delta L - \frac{2\delta}{\omega}\frac{\partial L}{\partial t}, \qquad (148)$$

the second order partial derivative of L being ignored.

The last dissipative term in equation (133) is induced by the component $\mu e\dot{v}'$; it depends on the tidal lag δ_c and applying approximation (134) may be presented in the form

$$\mu e\dot{v}' = \mu e\dot{v} - \mu\delta_c\frac{e}{\omega}\ddot{v}. \qquad (149)$$

The second order time derivative $\ddot{v}$ at the right part may be evaluated in virtue of the simplified differential equation (139), (140) from which relations (141) and (143) have been derived. Making use of these relations we have

$$\ddot{v} = -v\omega^2 + 2i\omega\left(1 - \frac{\nu}{\alpha}\right)L.$$

Inserting this expression for $\ddot{v}$ to the right part of relation (149), and then excluding $\dot{v}$ with the help of equation (141), expression (149) becomes:

$$\mu ie\omega v' = \mu ie\omega v\left(1 - \frac{\nu}{\alpha}\right)L + \delta_c e\mu\left[\omega v - 2i\left(1 - \frac{\nu}{\alpha}\right)L\right]. \qquad (150)$$

At last applying relations (147), (148) and (150), differential equation (133) for v is expressed in the form that explicitly accounts the dissipation in the fluid core:

$$\dot{u} + \dot{v} + iv\omega\left[1 + e_c - \frac{\mu e}{\alpha}(1 + i\delta_c)\right] - \frac{\nu}{\alpha}\left[L\left(1 - \frac{2}{3}e\right) + \frac{i}{\omega}\frac{\partial L}{\partial t}\right] +$$

$$+\frac{\nu\delta}{\alpha}\left[-iL(1-e) + \left(\frac{2}{\omega}\right)\frac{\partial L}{\partial t}\right] = 0 \qquad (151)$$

3.7. Equations of the Earth's rotation in the inertial frame

It is naturally to study precession and nutation in the ecliptical inertial coordinate frame. With this aim let us transform the fast variables u, v that vary with near diurnal frequencies to the slow variables D, Q in the following way:

$$D = u\exp(i\psi),\;\; Q = v\exp(i\psi), \tag{152}$$

where $D = \dot{\theta} + i\dot{\phi}\sin\phi$ as a consequence of the Euler's kinematic equations (19).

Let $\hat{\overline{\rho}} = (\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ be coordinates of the unit vector to the perturbing body in the nonrotating equatorial system, the axis $\hat{\rho}_1$ coming along the node of the instant equator on the ecliptic, and let $\hat{\xi} = \hat{\rho}_1 + i\hat{\rho}_2, \hat{\zeta} = \hat{\rho}_3$ be the complex variables. Then ξ, ζ are connected with $\hat{\xi}, \hat{\zeta}$ by the relation

$$\hat{\xi} = \xi\exp(i\psi),\;\; \hat{\zeta} = \zeta, \tag{153}$$

and expression (73) for the rigid body torque L may be written as

$$L = \hat{L}\exp(-i\psi), \tag{154}$$

where

$$\hat{L} = -2ip\omega\hat{\xi}\hat{\zeta} \tag{155}$$

and so $\hat{L}$ does not depend on the rotational angle ψ.

In the variables D, Q differential equations (144), (145) and (151) have the form

$$\dot{D}\left(1 + \frac{2}{3}e\sigma\right) + \dot{Q}\left(\alpha + \frac{2}{3}e\nu\right) - i\omega D\left[1 + e\left(1 - \frac{1}{3}\sigma\right)\right] + iQ\nu K = \\ = L_u, \tag{156}$$

$$\dot{D} + \dot{Q} - i\omega D + i\omega Q\left[e_c - \frac{\mu}{\alpha}e(1 + i\delta_c)\right] = L_v, \tag{157}$$

where the parameter K is defined by equality (145), the right parts L_u, L_v are given by the expressions

$$L_u = \hat{L} + (\delta + i)\frac{\sigma}{\omega}\frac{\partial\hat{L}}{\partial t} + \hat{L}^d + \hat{L}^d_c, \tag{158}$$

$$L_v = \frac{\nu}{\alpha}\left[\hat{L}\left(1 - \frac{2}{3}e\right) + i\delta\hat{L}(1 - e) + \frac{i}{\omega}\frac{\partial\hat{L}}{\partial t} - \delta\left(\frac{2}{\omega}\right)\frac{\partial\hat{L}}{\partial t}\right], \tag{159}$$

the torques $\hat{L}^d, \hat{L}^d_c$ adsorb the dissipative terms omitted by the method of transfer function, and $\hat{L}$ is related to the rigid body torque L by equations (153)-(155). The torques $\hat{L}^d, \hat{L}^d_c$ will be obtained from L^d, L^d_c, given by expressions (107)–(109) and (146) respectively, if one replaces ξ, ζ by $\hat{\xi}, \hat{\zeta}$, and L by $\hat{L}$:

$$\hat{L}^d = -4 \sum_{k=1,2} p_k \epsilon_k \sigma \delta \left[\omega \hat{\xi}_k \hat{\zeta}_k + i \left(\hat{\zeta}_k \frac{\partial}{\partial t} \hat{\xi}_k - \hat{\xi}_k \frac{\partial}{\partial t} \hat{\zeta}_k \right) \right] \quad (160)$$

$$\hat{L}_c^d = \nu \delta_c \left[\frac{1}{2} pv (3\cos^2\theta - 2\cos\theta - 1) + ie(\alpha - \nu) \hat{L} \right] \quad (161)$$

Note that the functions $\hat{L}_u, \hat{L}_v$ do not depend on the rotational angle ψ.

4. Applications

4.1. Precession and obliquity rate

In order to calculate the precession and the obliquity rate we have to set $\dot{D} = \dot{Q} = 0$ in differential equations (156), (157) and average all the coefficients which depend on coordinates of the perturbing bodies. The reduced differential equations have the form

$$-i\omega D \left[1 + e \left(1 - \frac{1}{3} \sigma \right) \right] + iQ\nu \langle K \rangle = \langle L_u \rangle, \quad (162)$$

$$-i\omega D + i\omega Q \left[e_c - \frac{\mu}{\alpha} e(1 + i\delta_c) \right] = \langle L_v \rangle, \quad (163)$$

where

$$\langle L_u \rangle = \langle \hat{L} \rangle + \langle \hat{L}^d \rangle + \langle \hat{L}_c^d \rangle,$$

$$\langle L_v \rangle = \frac{\nu}{\alpha} \langle \hat{L} \rangle \left[1 - \frac{2}{3} e + i\delta(1 - e) \right].$$

Because we are going to calculate small perturbations due to non-rigidity of the Earth, the eccentricities as well as the inclinations to ecliptic of the lunar and solar orbits may be disregarded when averaging coefficients of these equations. Making use of the expressions given in Appendix for the averaged values of the arising combinations of the coordinates we obtain

$$\langle L \rangle = -\omega p \sin\theta \cos\theta,$$

$$\langle L^d \rangle = -2i\sigma\omega\delta \sin\theta \sum_{k=1,2} p_k \epsilon_k \left(\cos\theta - 2\frac{n_k}{\omega} \right),$$

$$\langle L_c^d \rangle = p\nu\delta_c \left[\frac{1}{2} Q (3\cos^2\theta - 2\cos\theta - 1) - ie\omega(\alpha - \nu) \right],$$

$$\langle \zeta_k^2 \rangle = \frac{1}{2} \sin^2\theta,$$

$$\langle K \rangle = \left(1 - \frac{3}{2} \sin^2\theta \right) p.$$

Ignoring dissipative terms, the solution for Q, D is as follows:

$$Q = -i\omega \frac{p}{f_c(1-\alpha)} \left(1 - \frac{\nu}{\alpha}\right) \sin\theta \cos\theta,$$
$$D = -ip \sin\theta \cos\theta + Q\nu \left(1 - \frac{3}{2}\sin^2\theta\right) p,$$

where the frequency f_c, defined by the expression

$$f_c = \frac{\omega}{1-\alpha}\left[e_c - \frac{\mu}{\alpha}e - \nu\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{p}{\omega}\right],$$

has meaning of the Free Core Nutation FCN frequency.

The last term of this expression for f_c is less than the others by two orders and may be disregarded. As $D = \dot{\theta} + i\sin\theta\dot{\phi}$ we obtain the following expression for the precessional rate $\dot{\phi}$

$$\dot{\phi} = -p\left(1 + \frac{dp}{p}\right)\cos\theta,$$

where dp is the contribution of the fluid core to the parameter p of the luni-solar precession:

$$\frac{dp}{p} = \frac{p}{f_c(1-\alpha)}\nu\left(1 - \frac{\nu}{\alpha}\right)\left(1 - \frac{3}{2}\sin^2\theta\right)$$

Adopting for f_c the numerical value that corresponds to the observed 431-day FCN period we have

$$\frac{p}{f_c} = 3.75'' = 1.82 \times 10^{-5},$$

and thus

$$dp = 11 \ \text{mas/cy}.$$

This effect is not obtainable by the method of transfer function but must be accounted when deriving the Earth's dynamical flattening from the observed value of the luni-solar precession.

From the geophysical point of view it seems very interesting to interpret the observed value $\dot{\theta}_{obs} = -24.08 \pm 0.017$ mas/cy of the obliquity rate based on VLBI data (see Shirai & Fukushima, 2001, ([16])). The main part of this value is not the dissipative effect but is the result of omission of some terms in the adopted rigid body nutation, as it has been at first shown in (Williams, 1994, [17]). In a more detail, these terms are due to so called 'tilt' effect of perturbations in the tilt of the lunar orbit to ecliptic (that gives −25.4 mas/cy), and due to direct perturbations from planets with the resulted contribution −1.4 mas/cy, the total effect being −26.8 mas/cy. The value by Williams may be compared with those given by the more recent rigid body models of nutation: −26.5 in SMART97 (Bretagnon et all. 1998, [1]) and -27.2 in RDAN97 (Roosbeek & Dehant, 1998, [14]). Thus the observed obliquity rate $\dot{\theta}_{obs}$ that has to be explained as the dissipative effect varies in the range:

$$\dot{\theta}_{obs} = 2.7 \div 3.1 \ \text{mas/cy}, \tag{164}$$

depending on the applied rigid body model.

With equations (162), (163) one can derive the following analytical expressions for the obliquity rate $\dot{\theta}_\delta$ induced by dissipation of the Earth as a whole and the rate $\dot{\theta}_{\delta_c}$ due to the dissipation caused by the differential rotation of the fluid core:

$$\dot{\theta}_\delta = 2p\sigma\delta \sin\theta(\epsilon\cos\theta - 2\tilde{\epsilon}), \tag{165}$$

$$\dot{\theta}_{\delta_c} = p\nu e(\alpha - \nu)\delta_c \sin\theta\cos\theta, \tag{166}$$

where

$$\epsilon = \frac{p_1\epsilon_1 + p_2\epsilon_2}{p} = 2.04 \times 10^{-5},$$

$$\tilde{\epsilon} = \frac{p_1 n_1 \epsilon_1 + p_2 n_2 \epsilon_2}{\omega p} = 6.27 \times 10^{-7}.$$

The component $\dot{\theta}_\delta$ of the obliquity rate may be reliably evaluated making use of the LLR estimate $\delta = 0.0376$ that gives the positive rates 0.675 mas/cy and 0.153 mas/cy as the impact of the Moon and Sun, relatively, with the total value 0.928 mas/cy. Then the remaining part of the observed obliquity rate (164) must be attributed to the effect of the fluid core:

$$\dot{\theta}_{\delta_c} = 1.8 \div 2.2 \ \text{mas/cy}.$$

Applying the theoretical expression (166) we obtain the estimate

$$\delta_c = 0.09 \div 0.11. \tag{167}$$

The large value of δ_c has been anticipated as the following reasoning seems to be plausible. The tidal lag δ of the Earth as a whole, obtained from LLR data, is the weighted sum of the contributions of the mantle and the fluid core. It is known that the dissipation in the mantle is weak and so its contribution to δ is small if any. Thus if only the core were responsible for the tidal lag δ then we could expect that the relation $\delta = \alpha\delta_c$ is valid which gives $\delta_c \approx 0.3$.

4.2. Transfer function

Let us reduce the derived above differential equations of the Earth's rotation restricting ourselves to the terms, perturbations from which may be calculated with the help of the method of transfer function. For that in the differential equations we retain only the terms which have the form either of the rigid body torque or its time derivative. In this way indeed it is possible to describe the most significant perturbations of rotation of the deformable Earth with the viscous fluid core (except the obliquity rate which cannot be modeled in this way; see considerations of the previous section).

Differential equations of the Earth's rotation reduced in this manner have the form:

$$\dot{u}\left(1+\frac{2}{3}e\sigma\right)-ie\omega(1-\sigma)u+\left(\alpha+\frac{2}{3}e\nu\right)(\dot{v}+i\omega v)=L_u, \tag{168}$$

$$\dot{u}+\dot{v}+iv\omega\left[1+e_c-\frac{\mu e}{\alpha}(1+i\delta_c)\right]=L_v, \tag{169}$$

where

$$L_u=L+(\delta+i)\frac{\sigma}{\omega}\frac{\partial L}{\partial t}+ie\nu\delta_c(\alpha-\nu)L, \tag{170}$$

and

$$L_v=\frac{\nu}{\alpha}\left[L\left(1-\frac{2}{3}e\right)+\frac{i}{\omega}\frac{\partial L}{\partial t}\right]-\frac{\nu\delta}{\alpha}\left[-iL(1-e)+\left(\frac{2}{\omega}\right)\frac{\partial L}{\partial t}\right].$$

With sufficient accuracy the right parts L_u, L_v of these equations may be considered as known functions of time after substituting the rigid body solution to them, i.e. setting $L=-i\omega(\dot{\theta}_0+i\dot{\phi}_0\sin\theta_0)\exp(-i\psi)$ and thus expressing the right parts in terms of the rigid body nutations. After that the equations reduce to a non-uniform system of linear differential equations and its solution may be obtained by applying a transfer function to each harmonics of the rigid body nutations. The omitted small additional perturbing terms are not of the such simple structure and cannot be expressed in terms of the rigid body nutations.

Unlike the SOS equations, on which the standard method of transfer function is based, equations (168), (169) explicitly depend on the two dissipative parameters δ, δ_c of the clear physical meaning; the parameter δ is the tidal phase lag of the Earth as a whole while δ_c is that of its fluid core, manifested by its differential rotation.

The prime interest of our study is to model the subtle effects of the Earth's anelasticity but not to accomplish a nutational series; so we restrict ourselves to demonstration that in the frame of this simplified model it appears possible to explain the observed values of the out-phase amplitudes of 18.6-year and semi-annual nutations and confirm the high degree of viscosity of the fluid core.

4.3. Out-phase nutation components

One of the most striking features of the nutations deduced from VLBI data is the large amplitudes of the observed out-phase terms. Their numerical values are reported in a number of works; all of these estimates are in accordance. In Table 2 the observed in-phase and out-phase amplitudes (in mas) are reproduced from the work (Shirai & Fukushima, 2001) ([16]) for the three main nutations (including the fortnightly nutation). In this table there are also given the corresponding estimates of δ_c, obtained with the help of the analytical relation deduced further in this section, and the derived from them values of the parameter μ^{dis} of damping of Free Core Nutation. Note that the amplitudes presented in Table 2 are not really observed quantities but the theory-dependent ones because they are obtained by the mentioned above formal method estimating the imaginary parts of the coefficients of the transfer function as solve-for parameters from which the "observed" out-phase amplitudes have been derived.

Table 2. Observed main nutations and estimates of δ_c

T (days)	$d\phi_{in}$	$d\phi_{out}$	δ_c	μ^{dis}	$d\theta_{in}$	$d\theta_{out}$	δ_c	μ^{dis}
6798.38	17206	3.3415	0.38	0.09	9205	-1.5062	0.47	0.11
182.62	-1317	-1.7168	0.46	0.11	579	-0.5702	0.44	0.10
13.66	-228	0.2857	-	-	98	0.1478	-	-

One can see that the four independent esimations of δ_c presented in Table 2 are in a good accordance. They are rather large and marginally out the boundary of the physically meaningful range of this value ($\delta_c < \delta/\alpha \approx 0.35$). The estimated values δ_c are very sensitive to the adopted values of the parameters ν, α and probably may be significantly diminished keeping reasonable values of them. From the other hand the large δ_c might be explained as a result of absorbing the effects of the ocean tides. And indeed, Table 12 in (Dehant & Defraigne, 1997, [2]) shows that contributions of the oceanic tides to the out-phase amplitudes of both the 18.6 year and semi-annual nutations are about half the observed values which means that only the half of the estimated δ_c is caused by the dissipation in the fluid core. The reduced value $\delta_c \approx 0.2$ becomes more consistent with the estimate $\delta_c = 0.10$ derived in the previous subsection from the observed obliquity rate (which parameter is not affected by the ocean tides).

Our efforts to apply the developed theory for explanation of the "observed" out-phase amplitudes of fortnightly mutation have failed as the predicted amplitudes are of the opposite signs (being approximately of the same absolute value). It is noteworthy that the observed fortnightly out-phase amplitudes are less by the factor 2 than the best post-fit rms 0.28 mas, reached in the work (Shirai & Fukushima, 2001, [16]). Because the formal consideration of the imaginary parts of the compliances is not based on a sound physical model, it is quite possible that the "observed" amplitudes of the fortnightly nutation is an artefact and at present namely this nutational component deteriorates quality of fitting to the observed positions of Celestial Pole. More reliable conclusions might be done after reprocessing these data with the developed model, estimating the parameter δ_c simultaneosly with ν, μ, and probably with e and e_c.

To derive analytical expressions for the out-phase amplitides we simplify equations (156), (157) setting $e = 0$ at the left parts of them, except the dissipative terms proportional to δ_c. Then equations (156), (157) in the variables D, Q (i.e. in the inertial frame) reduce to the form

$$\dot{D} + \dot{Q}\alpha - i\omega D = \hat{L}\left[1 + ie\nu(\alpha - \nu)\delta_c\right] + (\delta + i)\frac{\sigma}{\omega}\frac{\partial \hat{L}}{\partial t}, \tag{171}$$

$$\dot{D} + \dot{Q} - i\omega D + i\omega Q\left[e_c - \frac{\mu}{\alpha}e(1 + i\delta_c)\right] =$$
$$= \frac{\nu}{\alpha}\left[\hat{L}(1 + i\delta) + (i - 2\delta)\frac{\partial}{\partial t}\frac{\hat{L}}{\omega}\right]. \tag{172}$$

Subtracting these equations the variables D, $\dot{D}$ are excluded; then dividing the both parts of the obtained differential equation by $1 - \alpha$, and defining the frequency f_c by relation

$$f_c = \omega\left(e_c - \frac{\mu e}{\alpha}\right)(1 - \alpha)^{-1}, \tag{173}$$

(f_c has meaning of the frequency of free oscillations of the fluid core in the non-rotating coordinate frame, so called Free Core Nutation, FCN) we obtain the differential equation for Q in the form:

$$\dot{Q} + iQf_c(1 - i\mu^{dis}) = -\frac{\hat{L}}{1 - \alpha}\left[1 - \frac{\nu}{\alpha}(1 + i\delta) + i\nu_c^{dis}\right] +$$
$$+ \frac{1}{(1 - \alpha)\omega}\frac{\partial \hat{L}}{\partial t}\left[i\left(\frac{\nu}{\alpha} - \sigma\right) - \delta\left(2\frac{\nu}{\alpha} + \sigma\right)\right], \tag{174}$$

where μ^{dis}, ν_c^{dis} stand for the two undimensional parameters:

$$\mu^{dis} = \mu\frac{\omega}{f_c}\frac{e}{\alpha}\delta_c, \tag{175}$$

$$\nu_c^{dis} = e\nu(\alpha - \nu)\delta_c. \tag{176}$$

These parameters describe two different types of the dissipative effects caused by viscousity of the fluid core: the parameter μ^{dis} characterizes the damping of Free Core Nutation while ν_c^{dis} is a constant of the dissipative coupling between the mantle and fluid core due to their differential rotation.

As a passing remark let us note that from equation (173) it follows that the tidal interaction diminishes the FCN frequency, calculated for the model of the

Earth with a rigid mantle and a fluid core (Poincare model), in analogous way as it diminishes the Euler's frequency of the free pole oscillations of the rigid Earth model to the Chandler frequency of the elastic Earth. To underline this similarity, equation (173) may be written in the form

$$f_c = \omega \frac{e_c}{1-\alpha}(1-\sigma_f), \tag{177}$$

where

$$\sigma_f = \frac{\mu e}{\alpha e_c} = 0.239 \tag{178}$$

is the analogue of the corresponding coefficient $\sigma = 0.320$ in equation (22) for the Chandler's frequency.

The parameter μ^{dis} of the FCN damping may be expressed in terms of σ_f, δ_c by the relation

$$\mu^{dis} = \sigma_f \frac{\omega}{f_c} e_c \delta_c \approx 0.263\delta_c,$$

used to calculate the values of μ^{dis} presented in Table 2.

For evaluating the small dissipative effects the derivative $\dot{D}$ in equation (171) may be ignored and we have

$$D = -i\alpha \frac{\dot{Q}}{\omega} + i\frac{\hat{L}}{\omega}\left(1 + i\nu_c^{dis}\right) + (i\delta - 1)\frac{\sigma}{\omega^2}\frac{\partial \hat{L}}{\partial t}.$$

Let $d\theta_0, d\phi_0$ be nutations in inclination and longitude in the rigid body approximation. Then the complex variable $D^0 = d\dot{\theta}_0 + i\sin\theta_0 d\dot{\phi}_0$ may be presented in the form of the trigonometric series:

$$D^0 = i\dot{\phi}_0 \sin\theta_0 + i\sum_f f(d\theta_0^f + i\sin\theta_0 d\phi_0^f)\exp(i\lambda_f), \tag{179}$$

where $\lambda_f = ft + \Phi_f$, f is a linear combination of the fundamental frequencies, Φ_f is the phase and $d\theta_0^f$, $d\phi_0^f$ are the nutational amplitudes for the frequency f, $\dot{\phi}_0$ is the luni-solar precession in longitude. A solution for the variable Q may be constructed in the similar trigonometric form

$$Q = Q_0 + \sum_{f\neq 0} Q_f \exp(i\lambda_f), \tag{180}$$

where Q_0 is a constant.

In the rigid body approximation from equation (171) it follows the relation

$$\hat{L} = -i\omega D^0,$$

combining which with equation (155) we obtain the expression

$$\hat{L}_0 = -p\sin\theta_0\cos\theta_0\omega = \dot{\phi}_0\sin\theta_0\omega$$

for the constant part of $\hat{L}$. Then substituting series (179), (180) to equation (174) we find the coefficient Q_f for the nutational frequency f:

$$\frac{Q_f}{\omega} = i\left(d\theta_0^f + i\sin\theta_0 d\phi_0^f\right)\frac{f}{f+f_c}\left(1 + i\mu^{dis}\frac{f_c}{f+f_c}\right)\times$$
$$\times\left[1 - \frac{\nu}{\alpha}(1+i\delta) + i\nu_c^{dis} - \frac{f}{\omega}\left(\frac{\nu}{\alpha} - \sigma\right) - i\delta\frac{f}{\omega}\left(2\frac{\nu}{\alpha} + \sigma\right)\right] \tag{181}$$

for $f \neq 0$ and

$$\frac{Q_0}{\omega} = -i\frac{p}{f_c(1-\alpha)}\sin\theta\cos\theta\left(1 - \frac{\nu}{\alpha} + i\nu_c^{dis}\right)\left(1 + i\mu^{dis}\right) \tag{182}$$

for $f = 0$.

Calculating the out-phase components, the terms proportional to f/ω at the right part of equation (181) may be ignored and as the result

$$\frac{Q_f}{\omega} = i\frac{f}{f+f_c}\left(d\theta_0^f + i\sin\theta_0 d\phi_0^f\right)\left[1 - \frac{\nu}{\alpha}(1+i\delta) + i\nu_c^{dis} + i\mu^{dis}\left(1 - \frac{\nu}{\alpha}\right)\right]. \tag{183}$$

Now substitute the solution for the variable Q, given by expressions (180), (183), to equation (171) in order to obtain corrected values $d\theta$, $d\phi$ of the rigid body nutations $d\theta_0, d\phi_0$. Making use of approximation (183) the corrections to nutational amplitudes may be presented in the form:

$$d\theta^f + i\sin\theta d\phi^f = \left(d\theta_0^f + i\sin\theta_0 d\phi_0^f\right)\left(R_f^{in} + iR_f^{out}\right), \tag{184}$$

where R_f^{in}, R_f^{out} are the factors of the in-phase and out-phase components, the coefficient R_f^{out} of our prime interest being given by the expression

$$R_f^{out} = \left[-\nu\delta + \nu_c^{dis}\alpha + \mu^{dis}(\alpha - \nu)\right]\frac{f}{f+f_c}(1-\alpha)^{-1}. \tag{185}$$

When evaluating the dissipative factor ν_c^{dis} in relation (185) we have taken the estimate $\nu_c^{dis} \approx 8\times 10^{-6}$ that follows from the given above analysis of the observed obliquity rate. With the LLR based tidal lag $\delta = 0.03767$ the first term at the right part of equation (185) is about 2×10^{-3} i.e. by two order larger. Thus with the accuracy sufficient for our aims the following approximation may be used:

$$
\begin{aligned}
d\theta^f &= -R_f^{out} \sin\theta_0 d\phi_0^f, &(186)\\
\sin\theta d\phi^f &= R_f^{out} d\theta_0^f, &(187)
\end{aligned}
$$

where

$$
R_f^{out} = \left[-\nu\delta + \sigma_f (\alpha - \nu)\delta_c\right] \frac{f}{f + f_c} (1 - \alpha)^{-1}, \tag{188}
$$

and the parameter σ_f is defined by relation (178).

The two additive terms at the right part of the expression for R_f^{out} contribute to R_f^{out} with opposite signs and in a large degree cancel each other. Only combining the action of the dissipation of the Earth as a whole (the δ term) with that of the fluid core (the δ_c term) the observable amplitudes of the out-phase components of the two main nutations may be theoretically explained.

These estimates of δ_c seem to be more reliable than that derived above from the secular obliquity rate (see (167)) which strongly depends on accuracy of calculations of the rigid body obliquity rate (the tilt effect). This rigid body effect discovered by Williams explains the basic part of the observed rate while the dissipation is responsible only for its small fraction which manifests itself as a discrepancy of the observed obliquity rate with the tilt effect. Contrary, as Williams has shown the planetary terms in the rigid body model contribute but slightly to the out-phase coefficients of the main nutations (on the level of 0.1 mas) and thus namely the dissipative effects of the fluid core and (as it has been in brief discussed above) of the ocean tides are responsible for the main part of the large observed values of these coefficients.

In the next section we consider other sources of information on the Earth's dynamical Love numbers and the tidal lag δ_c.

4.4. Near-diurnal effects in site displacements and in the tesseral harmonics c_2^1, s_1^2 of the geopotential

Besides the considered above perturbations of the Earth's rotation, the differential rotation of the fluid core produces the following two observable effects:

1. Near-diurnal site displacements. The most significant of them is the K_1 tidal constituent with the period of one sidereal day. In the IERS standards ([11]) for the amplitude of the vertical site displacement of the tidal wave K_1 the value 12.5 mm is adopted (at the latitude 45 degree).

2. Near-diurnal oscillations of the tesseral harmonics c_2^1, s_2^1 with the largest amplitude of the K_1 component. In the IERS standards the amplitude of the K_1 tide in the normalized tesseral harmonics is given as $d\bar{c}_2^1 = d\bar{s}_2^1 = 472.0 \times 10^{-12}$.

Both these effects implicitely depend on the dynamical Love number k_2^d as they are proportional to the differential rotation v of the fluid ruled by differential equations of the Earth's rotation (168), (169) which depend on k_2^d. The site displacements involve also an explicit dependence on the dynamical Love numbers h_2^d and l_2^d through the scaling factors in the corresponding expressions for the vertical and transveral components of the displacements, while the scaling factor in the expression for the variations of the coefficients c_2^1, s_2^1 depends only on the dynamical Love number k_2^d. That is why the second effect is more informative for study of the Earth's rotation and may be used to estimate some of the parameters of the Earth's rotation independently on the VLBI data.

If the tides caused by the differential rotation of the fluid core are not taken into consideration in the explicit way, these effects may be described in terms of frequency dependence of the Love numbers. Such an alternative formulation is used in the work ([10])) which is the theoretical base of the corresponding sections of the IERS standards. It seems useful to present explicit analytical expressions for both these effects because they are much more simple that those given in IERS standards. In our work (Krasinsky, 2002, [7]) such expressions have been presented but only for the Poincare's model of the Earth's rotation. Now we see that the accuracy of this model is not sufficient, though the modeled corrections are small. For brevity we shall consider here only the main effect of the K_1 tide. In fact the difference with the Poincare's model is a modification of the scale factor which is the same for any of the tidal components; thus the expressions given in paper ([6]) may be easily modified.

Let us consider the K_1 tide in the site displacement. Its amplitude is proportional to the centrifigal potential W_f of the differential rotiation of the fluid core presented by the two last terms of the centrifugal potential W_r given (113):

$$W_f = -2(v_1 x_1 + v_2 x_2) x_3 \omega,$$

where x_1, x_2, x_3 are coordinates of the site on the Earth's surface.

In accordance with the general theory (see for instance (Moritz & Mueller, 1987, [12])) the vertical site displacement H, induced by the perturbing tidal potential W_f, may be presented by the expression

$$H = h_2^d R \frac{W_f}{W_0},$$

where

$$W_0 = \frac{G m_E}{R}$$

is the unperturbed Earth's potential on the Earth's surface of the main radius R.

If φ, λ are the corresponding latitude and longitude of the site, its displacement H may be given in the form

$$H = -Rh_2^d \epsilon \sin 2\varphi (v_1 \cos\lambda + v_2 \sin\lambda),$$

where ϵ is the undimensional constant:

$$\epsilon = \frac{R^3\omega^2}{Gm_E} = 0.003461. \tag{189}$$

Making use of the secular Love number k_s defined by relation (15), the coefficient ϵ may be expressed in terms of k_s and the zonal coefficient J_2 of the geopotential:

$$\epsilon = 3\frac{J_2}{k_s}. \tag{190}$$

To calculate the displacement caused by the K_1 tide, only the main component has to be retained in the solution for v; it has the form $v = \exp(-i\psi)Q_0$, where the constant Q_0 is presented by expression (182). Then for the components v_1, v_2 we can write

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{p}{f_c(1-\alpha)} \sin\theta\cos\theta \left(1 - \frac{\nu}{\alpha}\right) \left[\begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix} + \mu^{dis} \begin{pmatrix} -\cos\psi \\ \sin\psi \end{pmatrix}\right] \tag{191}$$

and inserting these relations to equation (189) the following expression for the vertical site displacement due to K_1 tide is obtained:

$$H = K_1^H R \sin\theta\cos\theta\sin 2\varphi \left[\sin(\lambda+\psi) - \mu^{dis}\cos(\lambda+\psi)\right]. \tag{192}$$

where

$$K_1^H = 3\frac{h_2^d}{k_s} J_2 \left(1 - \frac{\nu}{\alpha}\right) \frac{p}{f_c(1-\alpha)}. \tag{193}$$

In these relations ψ is the rotational angle counted from the ascending node of equator on ecliptic while the Greenwich Sidereal Time s is the same angle counted from the ascending node of ecliptic on equator; thus the relation

$$\psi = s + \pi$$

holds true.

Expression (192) coincides in its form with the amplitude of the K_1 tidal wave in the vertical site displacement presented in IERS Conventions (McCarthy, 2000, [11]) in terms of frequency dependence of the Love number h_2 (with the Doodson's notation $\tau = s + \pi$ used for the argument ϕ). In our analysis of VLBI observations of NEOS-A program 1998-2001 presented in (Krasinsky, 2002, [7]) the estimation 19.9±0.5 mm has been obtained for the numerical value of the coefficient in this relation (which estimate is to be compared with the theoretical value 12.00 mm given by the IERS Conventions of 2000, [11]). Applying the theoretical relation (192) we have the estimate $h_2^d = 0.23 \pm 0.01$ instead of $h_2^d = 0.095 \pm 0.005$ obtained in (Krasinsky, 2002, [7]) for the Poincare's model of the Earth's rotation.

Now let us consider variations of the coefficients c_2^1, s_2^1 of the geopotential due to the K_1 tide. They depend on the additional potential dW_f (114) induced by the Earth's deformations due to the differental rotation of the fluid core. The sign of the potential dW_f given by expression (114) must be reverted as dW_f in that form has been used to calculate the action of the outer bodies onto the Earth's rotation while now we calculate the opposite action of the tidally deformed Earth onto the outer point. Besides, the mass m in expression (114) must be set equal to unit. The potential dW_f has to be added to the geopotential W written in the standard form

$$W = \frac{Gm}{r} \sum_{k,j} \left(\frac{R}{r}\right)^k P_k^j(\cos\varphi)(c_k^j \cos j\lambda + s_k^j \sin j\lambda), \tag{194}$$

where φ, λ are the latitude and longitude of the probing point, P_k^j are associated Legendre polynomials. In order to present potential (114) in the similar form the combinations $\rho_1\rho_3, \rho_2\rho_3$ also must be expressed in terms of the associated Legendre polynomials:

$$\begin{aligned} \rho_1\rho_3 &= \cos\varphi\sin\varphi\cos\lambda = \frac{1}{3}P_2^1(\cos\varphi)\cos\lambda, \\ \rho_2\rho_3 &= \cos\varphi\sin\varphi\sin\lambda = \frac{1}{3}P_2^1(\cos\varphi)\sin\lambda. \end{aligned}$$

Then we obtain

$$dW_f = -\frac{R^2}{3r^3}P_2^1(\cos\varphi)(v_1\cos\lambda + v_2\sin\lambda).$$

Thus accounting for the additional potential dW_f we have the following corrections dc_2^1, ds_2^1 to be added to the coefficients c_2^1, s_2^1:

$$\begin{pmatrix} dc_2^1 \\ ds_2^1 \end{pmatrix} = \frac{\epsilon}{3}k_2^d \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

with the coefficient ϵ given by equation (189).

For the largest constituent K_1 we use expression (191) for v_1, v_2; then replacing k_2^d by the relation $k_2^d = \nu k_s$, making use of relation (190), we obtain

$$\begin{pmatrix} dc_2^1 \\ ds_2^1 \end{pmatrix}_{K_1} = K_1 \sin\theta \cos\theta \left[\begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix} + \mu^{dis} \begin{pmatrix} -\cos\psi \\ \sin\psi \end{pmatrix} \right], \tag{195}$$

where

$$K_1 \equiv J_2 \nu \left(1 - \frac{\nu}{\alpha}\right) \frac{p}{f_c(1-\alpha)}. \tag{196}$$

Formula (195) is of the same structure as the corresponding one in the IERS standards (after replacing ϕ by $s + \pi$) but has the advantage presenting the scale factor K_1, as well as the out-phase amplitude, in terms of the well defined constants which also enter the differential equations of the Earth's rotation. With the adopted values of these constants in relations (196) we can evaluate the scale factor K_1:

$$K_1 = 502 \times 10^{-12}.$$

For comparison with the IERS standards, the coefficient K_1 must be multiplied by the factor $\sqrt{5/3}$ to obtain corrections to the normalized coefficients $\bar{c}_2^1, \bar{s}_2^1$. The numerical value of the corresponding scale factor $\overline{K}_1$ is as follows:

$$\overline{K}_1 = K_1 \sqrt{5/3} = 646 \times 10^{-12} \tag{197}$$

while in the IERS standards ([11]) $\overline{K}_1 = 471.8 \times 10^{-12}$.

Thus one can see that the value of $\overline{K}$ adopted as the IERS Conventions of 2000 is inconsistent with the conventional theory of the Earth's rotation. Quite recently a successful attempt to estimate the amplitude $\overline{K}$ from laser ranging to the satellites Etalon-1 and Etalon-2 (1999-2002) has been carried out by Ivanova & Shuygina [4]. The result appeared to be in excellent accordance with the theoretical prediction (197):

$$\overline{K}_1 = (635 \pm 81) \times 10^{-12}. \tag{198}$$

It is interesting that the ratio of the out-phase and in-phase amplitudes of the K_1 tide in IERS standards is equal to 0.068 and qualitatively agrees with the value $\mu^{dis} \approx 0.10$ derived from the analysis of the out-phase nutation amplitudes (see Table 2). The out-phase amplitude of the K_1 tide in IERS standards is a result of modeling of ocean tide effects. Its lesser value in comparison with our estimate of μ^{dis} probably confirms that the tidal lag δ_c of the fluid core derived above from the analysis of the out-phase nutation amplitudes is an effective value that includes the contribution from the oceanic tides (about 50 %).

From the other hand the imaginary component of the FCN frequency obtained in (Shirai & Fukushima, 2001, [16]) being devided by the FCN frequency gives the lesser estimate $\mu^{dis} = 0.014$. However it may not be considered as a serious contradiction because the physical meaning of the parameters estimated by the semi-empirical method is not evident.

It seems that the out-phase amplitude of the K_1 tide (and thus the dumping parameter μ^{dis}) might be estimated experimentally simultaneously with its in-phase amplitude while processing the SLR data. Indeed the error of estimate (198) probably can be decreased including observations of Lageos and refining the procedure of the processing. In principle μ^{dis} might be estimated also with the help of VLBI observations from study of the vertical out-phase displacements of the K_1 tides. However SLR data are more sensitive because the considered effects in orbital elements of satellites accumulate with time.

The experimental estimations of the FCN damping parameter μ^{dis}, deduced from analysis of the SLR data, would be very important for understanding the geophysics behind the Earth's rotation.

5. Appendix

Appendix 1

Averaging

Connection of the body fixed equatorial coordinates (ρ_1, ρ_2, ρ_3) of the unit vector $\overline{\rho}$ to the perturbing body with corresponding non-rotating eqautorial coordinates $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ related to the equator of date is as it follows:

$$\begin{aligned} \rho_1 &= \hat{\rho}_1 \cos\psi + \hat{\rho}_2 \sin\psi, \\ \rho_2 &= -\hat{\rho}_1 \sin\psi + \hat{\rho}_2 \cos\psi, \\ \rho_3 &= \hat{\rho}_3. \end{aligned}$$

Averaging the combinations ρ_1^2, ρ_2^2 relatively to the rotational angle ψ we obtain

$$\begin{aligned} &\langle\rho_1^2\rangle = \langle\rho_1^2\rangle = \frac{1}{2}\left(\rho_1^2 + \rho_2^2\right) = \frac{1}{2}\left(1 - \rho_3^2\right), \\ &\langle\dot{\rho}_1\rho_1\rangle = \langle\dot{\rho}_2\rho_2\rangle = 0, \end{aligned}$$

that justifies relations (117) used in Section 3.3.

Connection of the equatorial coordinates $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ with the inertial ecliptical coordinates $(\rho_1^e, \rho_2^e, \hat{\rho}_3^e)$ has the form:

$$\begin{aligned} \hat{\rho}_1 &= \rho_1^e \cos\phi + \rho_2^e \sin\phi, \\ \hat{\rho}_2 &= -\xi \sin\phi \cos\theta + \eta \cos\phi \cos\theta + \zeta \sin\theta, \end{aligned}$$

$$\hat{\rho}_3 = \rho_1^e \sin\phi \sin\theta - \rho_2^e \cos\phi \sin\theta + \zeta \cos\theta.$$

Approximating the motion of the perturbing body by a circular ecliptical orbit we have:

$$\rho_1^e = \cos\Lambda, \;\; \rho_2^e = \sin\Lambda, \;\; \rho_3^e = 0,$$

where Λ is the mean longitude of the body under consideration, thus

$$\begin{aligned}
\hat{\rho}_1 &= \cos(\Lambda - \phi), \\
\hat{\rho}_2 &= \sin(\Lambda - \phi)\cos\theta, \\
\hat{\rho}_3 &= -\sin(\Lambda - \phi)\sin\theta,
\end{aligned}$$

and averaging the needed combinations of the coordinates relatively to the mean longitudes of the perturbing body we obtain

$$\begin{aligned}
\langle \hat{\rho}_1 \hat{\rho}_3 \rangle &= 0, \\
\langle \hat{\rho}_2 \hat{\rho}_3 \rangle &= -\frac{1}{2}\sin\theta\cos\theta, \\
\langle \hat{\rho}_3^2 \rangle &= \frac{1}{2}\sin^2\theta,
\end{aligned}$$

that justifies relations (135) and (137) used in Section 3.6 as well as those of Section 4.1.

Appendix 2

Oppolzer's corrections

Equations of the Earth's rotation (23)-(28) describe time behavior of the Euler's angles related to the main axes of inertia. If these angles were used to transform Earth fixed coordinates of stations to the inertial coordinate frame then it would be necessary to fulfill a correction for forced oscillations of the terrestrial poles applying so called Oppolzers's terms of near diurnal frequencies. It appears that such reduction may be avoided by referring the Euler's angles not to the equator of the Earth's figure but to the plane orthogonal to the instant angular velocity $\overline{\omega}$. Such approach is adopted in IERS standards and in order to be consistent with them the angles θ, ϕ obtained by integrations of equations (23)-(28) must be corrected in a special way. Freely speaking we shall use the term of Oppolzer's corrections to the nutation. Taking in mind that the derived above eqiations of the Earth's rotation is proposed to integrate numerically, it seems useful to give explicit expressions for the Oppolzer's corrections to be applied to the nutational angles θ, ϕ obtained by the numerical integration.

Let $\overline{\omega}^e = (\omega_1^e,\ \omega_2^e,\ \omega_3^e)$ be the column of coordinates of the angular velocity vector in the inertial coordinate system. It is connected with the coordinates $\overline{\omega}$ in the body fixed coordinates by the relation:

$$\overline{\omega}^e = P_3(-\phi)P_1(-\theta)P_3(-\psi)\overline{\omega}. \tag{199}$$

The coordinates $\omega_1,\ \omega_2,\ \omega_3$ of the angular velocity $\overline{\omega}$ may be expressed in terms of the time derivatives $\dot{\theta},\ \dot{\phi}$ by reversing the Euler's kinematic equations (19):

$$\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}. \end{aligned} \tag{200}$$

Inserting the values $\dot{\theta},\ \dot{\psi}$ obtained by integration of equations of the Earth's rotation to the right part of equations (200) the forced oscillation of the angular velocity in the body fixed coordinate frame may be obtained. Then the corresponding pole coordinates $m_1 = \omega_1/\omega,\ m_2 = \omega_2/\omega$ are presented as a sum of periodic components of nearly diurnal periods (so called Oppolzer's terms). In accordance with the adopted approach they have to be eliminated by redefining the nutational angles. Then the near diurnal terms in $\overline{\omega}$ are transformed to corrections to the nutational coefficients of the fundamental nutational arguments. To derive these corrections we transform the coordinates of the instant angular velocity $\overline{\omega}$ given by Equations (200) from the rotating frame to the inertial system making use of Equation (199). Neglecting the products of $\dot{\theta}$ and $\dot{\psi}$ we have after simple transformations:

$$\begin{pmatrix} \omega_1^e \\ \omega_2^e \\ \omega_3^e \end{pmatrix} = \omega_3 \begin{pmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} + \begin{pmatrix} \dot{\theta}\cos\phi - \dot{\phi}\sin\phi\sin\theta\cos\theta \\ \dot{\theta}\sin\phi + \dot{\phi}\cos\phi\sin\theta\cos\theta \\ \dot{\phi}\sin\theta \end{pmatrix}.$$

Let $\theta',\ \phi'$ are the angles of the nutation and precession of the instant equator; then the relations for $\overline{\omega}'$ may be written in the form:

$$\begin{pmatrix} \omega_1^e \\ \omega_2^e \\ \omega_3^e \end{pmatrix} \equiv \omega \begin{pmatrix} \sin\theta'\sin\phi' \\ -\sin\theta'\cos\phi' \\ \cos\theta' \end{pmatrix} = \begin{pmatrix} \sin(\theta + d\theta)\sin(\phi + d\phi) \\ -\sin(\theta + d\theta)\cos(\phi + d\phi) \\ \cos(\theta + d\theta) \end{pmatrix}.$$

If $\theta',\ \phi'$ are presented as

$$\begin{aligned} \theta' &= \theta + d\theta, \\ \phi' &= \rho + d\phi \end{aligned}$$

then comparing these two presentation of ω_1^e, ω_2^e, ω_3^e (neglecting the difference between ω_3 and ω) we obtain for $d\theta$, $d\phi$:

$$d\phi = \frac{\dot{\theta}}{\omega}\left(\frac{1}{\sin\theta}\right), \tag{201}$$

$$d\theta = -\frac{\dot{\phi}}{\omega}\sin\theta. \tag{202}$$

Thus the Oppolzer's corrections to the instant pole of rotation in the body fixed frame will be taken into account if after integration the nutational angles ϕ, θ are corrected by the additive terms $d\theta$, $d\psi$ given by the relations (201), (202).

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