IAA Transactions, No. 8, "Celestial Mechanics", 2002

On the detection of slow diffusion along resonances in Hamiltonian systems

E. Lega[1], **C. Froeschlé**[1], **M. Guzzo**[2]

[1]Observatoire de Nice, Bv. de l'Observatoire, B.P. 4229, 06304 Nice cedex 4, France
[2] Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Belzoni 7, 35131 Padova, Italy

It is now well known that deterministic systems can give rise to so–called chaotic motion [13]. Sometimes, there has been the tendency to associate chaotic motion with unstable motion. Yet many examples have been provided in the literature of chaotic motions which seem to remain stable up to very long times (see Benettin et al., 1985) [11].

Such behavior, detected in different fields of physics (beam–beam interaction, asteroidal motion), is now known to be typical of a certain class of dynamical systems.

Indeed, the representation of resonant motions given in (Benettin and Gallavotti, 1986) in the framework of the stability result of Nekhoroshev (Nekhoroshev, 1977) [20] shows that in quasi–integrable Hamiltonian systems there typically exist resonant chaotic motions whose actions are bounded up to an exponentially long time. Morbidelli and Froeschlé (Morbidelli and Froeschlé, 1996) [18] have shown, using a quite simple model, that the actions can remain confined up to very long times despite the fact that the largest Lyapunov characteristic exponent associated to the motion is quite large.

The existence of diffusive chaotic orbits has been heuristically shown in (Chirikov, 1979) [5] as due to the overlapping of resonances. Let us remark that such diffusion can be quite slow when the harmonics of the overlapping resonances are small (see for example Morbidelli and Guzzo, 1997) [19]. Therefore, such a slow diffusion is not easily recognized using even very long numerical integrations looking at the variations of the actions. It is very difficult to distinguish between the two different regimes with purely analytic tools although many improvements in this direction have been recently obtained (Celletti and Chierchia, 1995, [2]; Celletti and Chierchia, 1997, [3]; Celletti et al. 2000, [4]; Locatelli and Giorgilli, 2000, [17]). Therefore, numerical tools have been developed in the last ten years

(Laskar, 1990, Laskar et al., 1992, Lega and Froeschlé, 1996, Contopoulos and Voglis, 1997, Froeschlé and Lega, 1998) [14], [15], [16], [6], [9] to investigate the problem in an indirect way, i.e. by looking for the mathematical consequences of the two regimes, which are different from the stability of the actions.

In a previous work we have used two tools recently introduced to investigate the transition from the Nekhoroshev to the Chirikov regime, both in Hamiltonian systems and symplectic maps. The first tool, called the Fast Lyapunov Indicator (hereafter called FLI, Froeschlé et al., 1997; Lega and Froeschlé, 1997) [10], [16] is related to the computation of the tangent map for a suitable choice of an initial tangent vector, and allows us to distinguish rapidly not only slow chaos from ordered motion (Froeschlé et al., 1997) but also discriminates between regular resonant motions and tori (Froeschlé et al., 2000; Froeschlé and Lega, 2000; Lega and Froeschlé, 2001) [1], [8], [7]. Indeed, such numerical studies show that the computation of the FLI on a grid of regularly spaced initial conditions permits detection of the geometry of resonances with quite short numerical integrations. Let us remark that the definition of the FLI is very close to that of the Finite Time Lyapunov Exponent (see for example Tang and Boozer, 1996). These two indicators, defined independently, differ mainly in the dependence on the choice of the initial tangent vector.

The second method, introduced in Guzzo and Benettin, 2001 [12], and called "analytically filtered Fourier analysis" (hereafter AFFA), is related to the representation of the spectrum of a generic observable for systems which are in the Nekhoroshev regime. It provides global information on the long–term stability properties of the system through computation of a few well chosen orbits. Of course this requires less CPU time than grid–based calculations. Using both methods we have measured an interval of transition, centerd on a given value of the perturbation parameter ϵ^*, from the Nekhoroshev to the Chirikov regime for a three degrees of freedom Hamiltonian system. The relationship between Nekhoroshev stability and diffusion as still to be explored numerically and this is the aim of the present work. We know from the Nekhoroshev theorem that the effective stability time is exponentially long with respect to the ratio ϵ^*/ϵ. This means that diffusion can in principle be detected if the system is close to the transition to the Chirikov regime, i.e. if $\epsilon \simeq \epsilon^*$. Using the FLI charts we have been able to select a set of resonant chaotic initial conditions for some values of ϵ lower than ϵ^* and to follow the motion of the corresponding orbits. We have observed diffusion along resonant lines, for decreasing value of ϵ up to $\epsilon \simeq \epsilon^*/10$. The measure of the diffusion coefficient as a function of the perturbation parameter seems to follow the expected exponential law.

References

1. Froeschlé C., Lega E. On the structure of symplectic mappings. The fast

Lyapunov indicator: a very sensitive tool. Cel. Mech. & Dyn. Astron., **78**, 167–195, 2000.

2. Celletti A., Chierchia L. A constructive theory of Lagrangian tori and computer-assisted applications. Dynamics Reported, New Series, Springer Verlag 4, 1995.

3. Celletti A., Chierchia L. On the stability of realistic three-body problems. Comm. Math. Phys., 1997, **186**, 413–449.

4. Celletti A., Giorgilli A., Locatelli U. Improved estimates on the existence of invariant tori for Hamiltonian systems. Nonlinearity, 2000, **13**, 397–412.

5. Chirikov B. V. An universal instability of many dimensional oscillator system. Phys. Rep., 1979, **52**, 263–379.

6. Contopoulos G., Voglis N. A fast method for distinguishing between order and chaotic orbits. Astron. Astrophys., 1997, **317**, 73–82.

7. Lega E., Froeschlé C. On the relationship between fast Lyapunov indicator and periodic orbits for symplectic mappings. Cel. Mech. & Dyn. Astron., 2001, **15**, 1–19.

8. Froeschlé C., Guzzo M., Lega E. Graphical evolution of the Arnold's web: from order to chaos. Science, 2000, **289**, 2108–2110.

9. Froeschlé C., Lega E. Twist angles: a fast method for distinguishing islands, tori and weak chaotic orbits. comparison with other methods of analysis. AA, 1998, **334**, 355–362.

10. Froeschlé C., Lega E., Gonczi R. Fast Lyapunov indicators: application to asteroidal motion. Cel. Mech. & Dyn. Astron., 1997a, **67**, 41–62.

11. Benettin G., Galgani L., Giorgilli A. Numerical investigations on a chain of weakly coupled rotatorsin the light of classical perturbation theory. Nuovo Cimento B, 1985, **89**, 103.

12. Guzzo M., Benettin G. A spectral formulation of the Nekhoroshev theorem and its relevance for numerical and experimental data analysis. Discrete and Continuous Dynamical Systems-Series B, 2001, **1**, 1–28.

13. Hénon M., Heiles C. The applicability of the third integral of motion: Some numerical experiments. A. J., 1964, **1**, 73–79.

14. Laskar J. The chaotic motion of the solar system. a numerical estimate of the size of the chaotic zones. Icarus, 1990, **88**, 266–291.

15. Laskar J., Froeschlé C., Celletti A. The measure of chaos by the numerical analysis of the fundamental frequencies. Application to the standard mapping. Physica D, 1992, **56**, 253.

16. Lega E., Froeschlé C. Fast Lyapunov Indicators. Comparison with other chaos indicators. Application to two and four dimensional maps. in The Dynamical Behaviour of our Planetary System., Kluwer Academ. Publ., J. Henrard and R. Dvorak eds., 1997.

17. Locatelli U., Giorgilli A. Invariant tori in the secular motions of the three–body palnetary system, preprint, 2000.

18. Morbidelli A., Froeschlé C. On the relationship between Lyapunov times and macroscopic instability times. Cel. Mech. & Dyn. Astron., 1996, **63**, 227–239.

19. Morbidelli A., Guzzo M. The Nekhoroshev theorem and the asteroid belt dynamical system. Cel. Mech. & Dynam. Astron., 1997, **65**, 107–136.

20. Nekhoroshev N. N. Exponential estimates of the stability time of near-integrable Hamiltonian systems. Russ. Math. Surveys, 1977, **32**, 1–65.