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# New precession formula

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We modified J. G. Williams' formulation of the precession and the nutation by using the 3-1-3-1 rotation [1] so as to express them in an arbitrary inertial frame of reference. It gives the precession-nutation matrix as the product of four rotation matrices as $\mathcal{NP} = \mathcal{R}_1(-\epsilon)\mathcal{R}_3(-\psi)\mathcal{R}_1(\varphi)\mathcal{R}_3(\gamma)$, and the precession one similarly as $\mathcal{P} = \mathcal{R}_1(-\bar{\epsilon})\mathcal{R}_3(-\bar{\psi})\mathcal{R}_1(\varphi)\mathcal{R}_3(\gamma)$. Here $\varphi$ and $\gamma$ are the angles to specify the location of the ecliptic pole of date in the given inertial frame, $\psi$ and $\bar{\psi}$ are the true and mean ecliptic angles of precession, respectively, and $\epsilon$ and $\bar{\epsilon}$ are the true and mean obliquities of the ecliptic, respectively. As a result, the pole coordinates of the true and mean equators are explicitly given in terms of the newly introduced precession angles. Although the expression of nutation matrix is unchanged, we recommend the usage of the above form of $\mathcal{NP}$ instead of preparing $\mathcal{P}$ and $\mathcal{N}$ separately because of faster evaluation. The formulation is robust in the sense it avoids a singularity caused by finite pole offsets near the epoch. Facing the singularity is inevitable in the current IAU formulation. By using a recent theory of the forced nutation of the non-rigid Earth, SF2001 [2], we converted the true pole offsets referred to the ICRF, observed by VLBI for 1979–2000, and compiled by USNO, to the offests in the above three angles of precession, $\bar{\psi}$, $\varphi$, and $\gamma$, while we fixed $\bar{\epsilon}$ as the combination of the linear part provided in SF2001 and the quadratic and higher terms derived by Williams (1994). From the converted offsets, we determined the best-fit polynomial expressions of the three precession angles in the ICRF by a weighted least square method where we kept the quadratic and cubic terms as the same as in Williams (1994). These constitute a new set of fundamental expressions of the precessional quantities. The combination of the new precession formula and the periodic part of SF2001 serves a good approximation of the precession-nutation matrix in the ICRF.

## 1. Introduction

It is well known that the combination of the IAU formula of precession [3] and the IAU theory of nutation [4] is incorrect at the level of 10 mas when referred to the International Celestial Reference Frame (ICRF). Apart from the periodic feature of the difference, which shows the error of the IAU nutation theory, we notice the existence of secular trends, especially that in longitude. Such secular departures exist mainly due to the error of the adopted polynomial coefficients of the IAU precession formula and partly due to the difference between the ICRF and the reference frame referred to the mean equator and equinox of the epoch, J2000.0.

There have been many studies on the revision of nutation theories. Refer the reports of IAU/IUGG Joint WG on this issue [5] and the attached comprehensive lists of the articles related. Most of these works treated the revision of precession in the form of linear correction terms added to their nutation theories. Then their results naturally lead to a possible revision of precession constant as given in the reports of IAU WG on Astronomical Standards [6]. See also Table 8 and Figure 5 of our previous work [2].

From a practical point of view, however, it is unsatisfactory to revise the precession constant only. Rather demanded is the replacement of the IAU precession formula as a whole. There was such an example [7]. Unfortunately their results gave no good agreement with the observation. On the other hand, there is a movement to bypass the concept of ecliptic and equinox in expressing the motion of the equatorial pole. At the last General Assembly of the IAU, this was adopted as the basic policy to establish the new IAU formulation to connect the ICRF and the International Terrestrial Reference Frame by way of an intermediate reference frame using the concept of the non-rotating origin. A formulation on this line was already given [8].

Anyway, what we need is a formulation predicting the secular motion of the equatorial pole in the ICRF such that its combination with a suitable theory computing the periodic feature provides satisfactory agreement with the observations. In this article, we report such an example created from an analysis of the VLBI observation mentioned.

## 2. Weakness of IAU Formulation

Now a considerable amount of the observed pole offsets are availlable. Then it would be thought easy to revise the IAU precession formula to fit to the obervation by updating the numerical values of its polynomial expressions. Unfortunately it is not so straightforward. When we plotted the corrections in the *true* equatorial precession angles, $(\delta\zeta, \delta\theta, \delta z)$, which are directly converted from the observed corrections in nutation, large departures appeared in $\delta\zeta$ near the epoch. This is clear from the differential relation $\delta\zeta \approx -\delta\epsilon/\sin\theta_{\mathrm{A}}$, which shows the possibility of divergence due to a small divisor, $\sin\theta_{\mathrm{A}}$. Of course, the periodic features in $\delta\theta$ and in the sum $\delta\zeta + \delta z$ reflect the periodic motion of the observed correction in nutation.

In order to reduce the magnitude of the periodic features and hopefully large departures in $\delta\zeta$ as well, we transformed the corrections from the true sense to the mean sense by replacing the IAU nutation theory with the periodic part[1] of a recent nutation theory [2]. The resulting corrections are the corrections in the mean equatorial precession angles, $(\delta\zeta_{\rm A}, \delta\theta_{\rm A}, \delta z_{\rm A})$. This time $\delta\theta_{\rm A}$ and the sum $\delta\zeta_{\rm A} + \delta z_{\rm A}$ are sufficiently small and are appropriate to be expressed as the corrections in polynomial of time. However, unchanged is the ill determination of $\delta\zeta_{\rm A}$ near the epoch although its magnitude is significantly smaller than the case of the true precession angles.

Thus the appearance of large departures in $\delta\zeta$ near the epoch hardly depends on the incorrectness of the nutation theory adopted in the conversion. This situation is the same as we face in the determination of Keplerian elements for a low inclination orbit. In that case, the longitude of node $\Omega$ and the argument of pericenter $\omega$ are ill-defined while the inclination $I$ and the longitude of pericenter $\varpi \equiv \Omega + \omega$ are well-defined. This phenomenon appears whether periodic perturbations are excluded or not. Therefore we should find another and robust way to express the precession matrix.

## 3. J. G. Williams' Formulation

Let us consider the cause of the ill determination we faced in the previous section. Careful examination of the procedure of the conversion reveals that the ill determination is caused by the fact that the ICRF does not satisfy the assumption of IAU formulation, namely the exact coincidence of the precession matrix and the unit matrix at the epoch, $\mathcal{P}_0 = \mathcal{I}$. To overcome this fragile property of the IAU formulation, we should adopt a new formulation which works well in an arbitrary inertial frame of reference.

Once J. G. Williams derived a new formulation by skipping the intermediate process of routing the ecliptic pole at the epoch, $\mathrm{C}_0$, and directly moving from the mean equatorial pole at the epoch, $\bar{\mathrm{P}}_0$, to the ecliptic pole of date, C, first [1]. In order to deal with non-zero pole offset at the epoch, we modify his formulation by replacing the starting point from $\bar{\mathrm{P}}_0$, to the $z$-axis of the given inertial frame, Z. As a result the precession is described as $\mathrm{Z} \rightarrow \mathrm{C} \rightarrow \bar{\mathrm{P}}$ while the precession-nutation is done similarly as $\mathrm{Z} \rightarrow \mathrm{C} \rightarrow \mathrm{P}$. In other words, we (1) first specify the ecliptic pole of date, C, in the given inertial reference frame and shift from the inertial frame to an ecliptic reference frame of date, (2) then specify the mean or true equatorial pole of date, $\bar{\mathrm{P}}$ or P, in the ecliptic reference frame and shift from it to the mean or true equatorial reference frame of date. Since two angles are necessary to specify the direction of a pole in the given reference frame, we need

[1] The contribution of FCN was omitted.

four rotational operations in total. Thus we express the precession matrix as

$$\mathcal{P}_{\mathrm{NEW}} \equiv \mathcal{R}_1\left(-\bar{\epsilon}\right)\mathcal{R}_3\left(-\bar{\psi}\right)\mathcal{R}_1(\varphi)\mathcal{R}_3(\gamma).$$

The new polar diagram leads to the following definition of these new angles as

$$\gamma \equiv 180^\circ - \angle\mathrm{YZC} = \angle\mathrm{CZ\hat{Y}},\ \varphi \equiv \overline{\mathrm{ZC}},\ \bar{\psi} \equiv \angle\mathrm{\bar{P}CZ},\ \bar{\epsilon} \equiv \overline{\mathrm{C\bar{P}}},$$

where Y and $\hat{\mathrm{Y}} \equiv -\mathrm{Y}$ denote the positive and negative $y$-axes of the given inertial frame of reference. Here we adopted different notations from Williams' one since the starting point is different. Also we assigned a different notaion $\bar{\epsilon}$ to the angle $\epsilon_{\mathrm{A}}$ so as to discriminate their realizations in polynomial forms. Since $\bar{\epsilon}$ is the mean obliquity of the ecliptic of date, we can use the same form in expressing the precession-nutation matrix as

$$(\mathcal{NP})_{\mathrm{NEW}} \equiv \mathcal{R}_1(-\epsilon)\mathcal{R}_3(-\psi)\mathcal{R}_1(\varphi)\mathcal{R}_3(\gamma)$$

where $\psi \equiv \bar{\psi} + \Delta\psi$, $\epsilon \equiv \bar{\epsilon} + \Delta\epsilon$. As Williams stressed, this unified treatment is a merit of the original Williams' formulation, which is inherited to the modified one. Also note that the usual nutations are used in this formulation. This is another merit. The recent formulation [8] to describe the equatorial pole coordinates, $X$ and $Y$, requires the preparation of a little different nutations, $\Delta_1\psi$ and $\Delta_1\epsilon$, the nutations referred to the eclptic *of the epoch*. They must be correctly converted from the existing theories of nutation giving the usual nutations, $\Delta\psi$ and $\Delta\epsilon$, the nutations referred to the ecliptic of date[2]. Further note that the number of rotational operations needed to express the precession-nutation matrix reduces from six of the IAU formulation to four of the new one. This is yet another merit of the new formulation.

## 4. Determination of Precession Angles

Now the pole offset at the epoch is sufficiently small as around $0.04''$, we approximate the new angles of precession to be almost the same as Williams' original precession angles as

$$\gamma \approx \xi_{\mathrm{A}},\ \varphi \approx \epsilon'_{\mathrm{A}},\ \bar{\psi} \approx \eta_{\mathrm{A}},\ \bar{\epsilon} \approx \epsilon_{\mathrm{A}}$$

whose numerical expressions are given in Table 5 of Williams (1994). Note that Williams' $\epsilon_{\mathrm{A}}$ is significantly different from that of IAU 1976 theory. In fact, we previously estimated $\bar{\epsilon}$ from the VLBI observations [2] as

$$\bar{\epsilon}_{\mathrm{SF}}(t) = 84\,381.442\,8 - 46.838\,8\,t$$

[2] Note that Eq.(15) of [8] describing the procedure of conversion is linear and ignores the second and higher order effects. The formula of rigorous transformation becomes tedious.

where the unit is arcsecond and $t$ is the time since J2000.0 measured in Julian centuries. Its linear part is quite close to that of Williams' $\epsilon_{\rm A}$. Adopting the second and higher order terms from Williams' result and keeping the cut-off level of the coefficients as 0.1mas at 1 century apart from the epoch, we fixed the polynomial expression of $\bar{\epsilon}$ as

$$\bar{\epsilon}(t) = 84\,381.442\,8 - 46.838\,8\,t - 0.000\,2t^2 + 0.002\,0t^3.$$

By using this formula for $\bar{\epsilon}$, we transformed the observed corrections in nutation to those of the new precession angles after removing the polynomial forms of their counterparts in Williams' expressions. Thus converted corrections in terms of the new precession angles distribute so smoothly as to be well approximated by polynomials of time.

In order to find the best polynomial approximation of the obtained corrections in the new precession angles, we executed the weighted linear least square method for linear functions. Quadratic and higher order fittings failed to be meaningful. Then the adopted solutions are linear as

$$\bar{\psi} - \eta_{\rm A} = -(0.0431 \pm 0.0006) + (0.0174 \pm 0.0100)\,t,$$

$$\varphi - \epsilon'_{\rm A} = (0.0389 \pm 0.0003) - (0.0044 \pm 0.0040)\,t,$$

$$\gamma - \xi_{\rm A} = -(0.0000 \pm 0.0000_1) - (0.0052 \pm 0.0000_2)\,t.$$

Note that the constant term of $\gamma - \xi_{\rm A}$, and therefore[3] that of $\gamma$, too, is practically equal to zero. This means that the ecliptic pole of the epoch precisely lies on the $yz$-plane of the ICRF. By adding the Williams' original expressions subtracted prior to the determination, we obtained the final results as

$$\bar{\psi}(t) = -0.043\,1 + 5\,038.473\,9\,t + 1.558\,4\,t^2 - 0.000\,2\,t^3,$$

$$\varphi(t) = 84\,381.447\,9 - 46.814\,0\,t + 0.051\,1\,t^2 + 0.000\,5\,t^3,$$

$$\gamma(t) = 10.552\,5\,t + 0.493\,2\,t^2 - 0.000\,3\,t^3,$$

where again we omit the terms not exceeding 0.1 mas in 1900–2100. We confirmed that small and of no sign of apparent secular trends are the residuals in nutations when the precession-nutation matrix is computed by the combination of the new precession formula whose coefficients are determined as above and the periodic part of the nutation theory of SF2001.

Note that the procedure to determine the polynomial forms of the new precession angles described here is applicable to any combination of the observation and the nutation theory as long as the latter gives the corrections to the IAU precession formula in polynomial form of the corrections in nutation. Thus the results presented here will be easily updated.

[3] Note that $\xi_{\rm A}(0) = 0$ by its definition.

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