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An analytical solution of Lagrange planetary equations valid also for very low eccentricities

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We give an analytical solution of the Lagrange planetary equations expressed in a set of non singular elements for eccentricity: the semi-major axis a, the inclination i, the ascending node Ω, $\lambda = \omega + M$ where ω denotes the argument of perigee and M the mean anomaly, and the elements $C = e \cos \omega$ and $S = e \sin \omega$ where e denotes the eccentricity.

The perturbations which can be computed are due to the gravity field of a planet, or to the influence of a third body, through an internal or an external potential. These potentials are expressed with respect to the former set of elements, and in particular use a formulation of the Kaula functions of eccentricity G (Wytrzyszczak, 1986) and H with respect to C and S.

The main study deals with the integration of the differential system verified by C and S, since the metric element e and the angular element ω are mixed. The differential system is divided into two parts: a first one which can be analytically integrated in an exact way, the second part being seen as a perturbation of the first one. More precisely, the first part of the differential system verified by C and S corresponds to a harmonic oscillator: our method takes into account, in a first step, long period variations of 4 elements: C, Ω, S, and λ.
It is therefore a generalization of the Kaula method(Kaula, 1966) expressed in 'classical' orbital elements, in which only the secular variations of the angles Ω, ω, M are computed during a first step.

Moreover, our method enlightens the behavior of the argument of perigee ω with time. Like (Cook, 1966), we show that its variations are not really linear with time, in particular for low eccentricities. We therefore use another angle, called θ, which is well defined even for eccentricities equal to zero. We use this angle θ to integrate the long as well as the short period variations of C and S, and as a consequence of all the other elements.

Here, we show why and how we have divided the differential system verified by a, C, i, Ω, S and λ in two components. The first part gives a long period solution a_1, C_1, i_1, Ω_1, S_1 and λ_1; the second part gives corrections Δa, ΔC, Δi, $\Delta\Omega$, ΔS and $\Delta\lambda$. There results: $a(t) = a_1(t) + \Delta a(t)$, $C(t) = C_1(t) + \Delta C(t)$, ...

We also show the way used to express Δa, ΔC, Δi, ΔS, ... in a synthetic formulation.

Like the method built by Kaula, our method is based on a development in powers of the eccentricity. But, there is no supplementary hypothesis induced by the expression of our theory in non singular elements for eccentricity: it can be used for bodies – artificial satellites or planets – with an eccentricity greater than or equal to zero.

Finally, we compare three methods in different dynamic configurations: this method, expressed in non singular elements for eccentricity (with an integration hypothesis based on θ), the method of Kaula, expressed in classical orbital elements (with an integration hypothesis based on ω), and the numerical integration. Tests show that, compared to the method of Kaula, our method is more accurate as soon as the eccentricity of the studied body is smaller than a few 10^{-2}.

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